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NUMERICAL METHOD FOR THE MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATIONS USING HYBRID LEGENDRE FUNCTIONS

S. Nemati[1], P. Lima[2], Y. Ordokhani[3]

[1] Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran
Babolsar, Iran
s.nemati@umz.ac.ir

[2] CEMAT-Departamento de Matematica, Instituto Superior Tecnico, UTL
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
plima@math.ist.utl.pt

[3] Department of Mathematics, Alzahra University
Tehran, Iran
ordokhani@alzahra.ac.ir

Abstract: A new method is proposed for the numerical solution of linear mixed Volterra-Fredholm integral equations in one space variable. The proposed numerical algorithm combines the trapezoidal rule, for the integration in time, with piecewise polynomial approximation, for the space discretization. We extend the method to nonlinear mixed Volterra-Fredholm integral equations. Finally, the method is tested on a number of problems and numerical results are given.

Keywords: mixed Volterra-Fredholm integral equations, hybrid Legendre functions, piecewise polynomial approximation, trapezoidal method

MSC: 65R20, 41A30, 65D30

1. Introduction

In this paper, we are concerned with the numerical solution of the linear mixed Volterra-Fredholm integral equations of the form

$$u(x,t) = f(x,t) + \int_0^t \int_0^a K(x,t,y,z)u(y,z)dydz, \qquad 0 \le x,y \le a,\ 0 \le z \le t \le T, \tag{1}$$

where $f(x,t)$ and $K(x,t,y,z)$ are given continuous real-valued functions defined on $[0,a]\times[0,T]$ and $\{(x,t,y,z):\ x,y\in[0,a],\ 0\le z\le t\le T\}$, respectively, and $u(x,t)$

is the unknown function to be determined. With this purpose, space discretisation is introduced, using a basis of hybrid Legendre functions, while time integration is performed using the trapezoidal rule. We will also consider an extension of the proposed method to nonlinear equations of the form

$$u(x,t) = f(x,t) + \int_0^t \int_0^a K(x,t,y,z)g(y,z,u(y,z))dydz,$$
$$0 \le x, y \le a, \ 0 \le z \le t \le T, \quad (2)$$

where g is nonlinear in u.

Various problems in physics, mechanics and biology lead to nonlinear mixed type Volterra-Fredholm integral equations. In particular, such equations appear in modeling of the spatio-temporal development of an epidemic, theory of parabolic initial-boundary value problems, population dynamics, and Fourier problems [2, 4, 8].

In its general form, a mixed Volterra-Fredholm integral equation can be written as

$$u(\mathbf{x},t) = f(\mathbf{x},t) + \int_0^t \int_\Omega K(\mathbf{x},t,\mathbf{y},z,u(\mathbf{y},z))d\mathbf{y}dz, \quad (3)$$

where $u(\mathbf{x},t)$ is an unknown real-valued function defined on $D = \Omega \times [0,T]$ and Ω is a closed subset of $\mathbb{R}^n$, $n = 1, \ 2, \ 3$. The functions $f(\mathbf{x},t)$ and $K(\mathbf{x},t,\mathbf{y},z,u)$ are given functions defined on D and $S = \{(\mathbf{x},t,\mathbf{y},z,u) : \ \mathbf{x},\mathbf{y} \in \Omega, \ 0 \le z \le t \le T\}$, respectively [2].

Different numerical methods have been applied to approximate the solution of equation (3) (see for example [1, 3, 5]).

In this paper we use hybrid Legendre and block-pulse functions to solve equations of the forms (1) and (2). Hybrid Legendre functions have been applied extensively for solving differential and integral equations and systems, and proved to be a useful mathematical tool. In [6], a basis of shifted Legendre functions has been applied to the numerical solution of nonlinear two-dimensional Volterra integral equations.

In comparison with the methods used previously to solve equation (3), the advantage of the present method is the high convergence rate, specially with respect to the space variable, which allows to obtain accurate results using small matrices and with a low computational effort (see numerical examples in Section 5). Together with its simple implementation, this makes the present algorithm an efficient tool for the solution of this type of equations.

The organization of the rest of the paper is as follows: In Section 2 hybrid Legendre functions and their basic properties are described. In Section 3 we describe the numerical method used to solve equation (1). In Section 4, the method is extended to solve a class of nonlinear mixed Volterra-Fredholm integral equations. Numerical results are reported in Section 5 and conclusions are presented in Section 6.

2. Properties of hybrid Legendre functions

2.1. Definition and function approximation

Hybrid functions $b_{ij}(x)$, for $i = 1, 2, \ldots, k$, $j = 0, 1, \ldots, M$ and $h = a/k$ are defined on the interval $[0, a)$ as

$$b_{ij}(x) = \begin{cases} L_j(2x/h - 2i + 1), & (i-1)h \le x < ih, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $L_j(x)$ denotes a Legendre polynomial of order j . Hybrid functions are orthogonal, since

$$\int_0^a b_{ij}(x) b_{mn}(x) = \begin{cases} h/(2j+1), & i = m \textit{ and } j = n, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Suppose that $V = L^2[0, a]$ and $\{b_{10}(x), b_{11}(x), \ldots, b_{kM}(x)\} \subset V$ is the set of hybrid Legendre functions and

$$B = \text{span}\{b_{10}(x), b_{11}(x), \ldots, b_{1M}(x), \ldots, b_{k0}(x), b_{k1}(x), \ldots, b_{kM}(x)\},$$

and $p(x)$ is an arbitrary element in V. Since B is a finite dimensional vector space, $p(x)$ has a unique best approximation $p_{k,M} \in B$, such that

$$\forall b \in B,\ \|p - p_{k,M}\|_2 \le \|p - b\|_2.$$

Since $p_{k,M} \in B$, there exist unique coefficients $p_{10}, p_{11}, \ldots, p_{kM}$ such that

$$p(x) \simeq p_{k,M}(x) = \sum_{i=1}^{k} \sum_{j=0}^{M} p_{ij} b_{ij}(x) = P^T \psi(x), \tag{5}$$

where

$$P = [p_{10}, \ldots, p_{1M}, p_{20}, \ldots, p_{2M}, \ldots, p_{k0}, \ldots, p_{kM}]^T, \tag{6}$$

and

$$\psi(x) = [b_{10}(x), \ldots, b_{1M}(x), b_{20}(x), \ldots, b_{2M}(x), \ldots, b_{k0}(x), \ldots, b_{kM}(x)]^T. \tag{7}$$

The hybrid coefficients p_{ij}, $i = 1, 2, \ldots, k$, $j = 0, 1, \ldots, M$ are obtained as

$$p_{ij} = \frac{2j+1}{h} \int_{(i-1)h}^{ih} p(x) b_{ij}(x) dx.$$

We now briefly describe a technique that will be used to integrate hybrid Legendre functions.

2.2. Operational matrix of dual

The integration of the product of two hybrid vectors satisfies [7]:

$$\int_0^a \psi(x)\psi^T(x)dx = D, \tag{8}$$

where D is a $k(M+1) \times k(M+1)$ matrix of the form

$$D = \begin{pmatrix} d & O & O & \dots & O \\ O & d & O & \dots & O \\ O & O & d & \dots & O \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & O & \dots & d \end{pmatrix},$$

in which O is the zero matrix of order $M+1$ and

$$d = h \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1/3 & 0 & \dots & 0 \\ 0 & 0 & 1/5 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1/(2M+1) \end{pmatrix}.$$

3. Numerical method

In this section we apply a numerical method using hybrid Legendre functions to the numerical solution of mixed Volterra-Fredholm integral equations of the form (1). With this purpose, we consider the time step size τ as

$$\tau = \frac{T}{N}.$$

Then the mesh nodes are defined by

$$t_0 = 0, \quad t_n = t_{n-1} + \tau, \quad n = 1, 2, \dots, N.$$

Collocating equation (1) in t_n, $n = 0, 1, \dots, N$, yields:

$$u(x, t_n) = f(x, t_n) + \int_0^{t_n} \int_0^a K(x, t_n, y, z)u(y, z)dydz. \tag{9}$$

Considering the notations $u^n(x) = u(x, t_n)$ and $f^n(x) = f(x, t_n)$ in (9), we have

$$u^0(x) = f^0(x),$$

$$u^n(x) = f^n(x) + \int_0^{t_n} \int_0^a K(x, t_n, y, z)u(y, z)dydz, \quad n = 1, 2, \dots, N. \tag{10}$$

Using the trapezoidal rule to perform the integration on z in (10) we obtain the approximation

$$u^n(x) \simeq f^n(x) + \frac{t_n}{2n}\int_0^a \Big(K(x,t_n,y,t_0)u^0(y) + K(x,t_n,y,t_n)u^n(y) + \\ + 2\sum_{i=1}^{n-1} K(x,t_n,y,t_i)u^i(y)\Big)dy. \tag{11}$$

Introducing the notation $K^{n,i}(x,y) = K(x,t_n,y,t_i)$ in (11), yields:

$$u^n(x) = f^n(x) + \frac{t_n}{2n}\int_0^a \Big(K^{n,0}(x,y)u^0(y) + K^{n,n}(x,y)u^n(y) + 2\sum_{i=1}^{n-1} K^{n,i}(x,y)u^i(y)\Big)dy. \tag{12}$$

We approximate the functions in (12) using the method described in the previous section as

$$u^i(x) \simeq u^i_{k,M}(x) = U_i^T\psi(x) = \psi^T(x)U_i, \tag{13}$$

$$f^n(x) \simeq f^n_{k,M}(x) = F_n^T\psi(x) = \psi^T(x)F_n \tag{14}$$

$$K^{n,i}(x,y) \simeq K^{n,i}_{k,M}(x,y) = \psi^T(x)\kappa_{n,i}\psi(y), \tag{15}$$

where U_n, $n = 1,\ 2,\ldots,N$, in (13) is the unknown vector, of order $k(M+1)$. Substituting approximations (13)–(15) into equation (12) and using the operational matrix of dual, we obtain

$$U_n = F_n + \frac{t_n}{2n}\left[\kappa_{n,0}DU_0 + \kappa_{n,n}DU_n + 2\sum_{i=1}^{n-1}\kappa_{n,i}DU_i\right],$$

which can be rewritten as

$$(I - \frac{t_n}{2n}\kappa_{n,n}D)U_n = F_n + \frac{t_n}{2n}\left[\kappa_{n,0}DU_0 + 2\sum_{i=1}^{n-1}\kappa_{n,i}DU_i\right], \quad n = 1,\ldots,N. \tag{16}$$

Equations (16) form a system of $k(M+1)$ linear equations in each step and can be solved easily using direct methods.
Therefore U_n, $n = 1,\ 2,\ldots,N$ can be computed via the recursive equation (16) using the initial value $U_0 = F_0$.

4. Numerical solution of nonlinear mixed Volterra-Fredholm integral equations

In this section we extend our numerical method to solve nonlinear mixed Volterra-Fredholm integral equations of the form (2).

Considering the same partition and notations as in Section 3 and collocating equation (2) in $t = t_n$ yields:

$$u^n(x) = f^n(x) + \int_0^{t_n} \int_0^a K(x, t_n, y, z) g(y, z, u(y, z)) dy dz. \tag{17}$$

Using the composite trapezoidal integration rule for the integral part of (17) leads to:

$$\begin{aligned} u^n(x) = f^n(x) + \frac{t_n}{2n} \int_0^a \Big(& K^{n,0}(x, y) g(y, t_0, u^0(y)) + K^{n,n}(x, y) g(y, t_n, u^n(y)) + \\ & + 2 \sum_{i=1}^{n-1} K^{n,i}(x, y) g(y, t_i, u^i(y)) \Big) dy. \end{aligned} \tag{18}$$

Introducing the notation $g^i(y) = g(y, t_i, u^i(y))$ equation (18) can be written as

$$u^n(x) = f^n(x) + \frac{t_n}{2n} \int_0^a \Big(K^{n,0}(x, y) g^0(y) + K^{n,n}(x, y) g^n(y) + 2 \sum_{i=1}^{n-1} K^{n,i}(x, y) g^i(y) \Big) dy. \tag{19}$$

We approximate the functions $u^i(x)$, $f^n(x)$ and $K^{n,i}(x, y)$ in equation (19) using (13)–(15) and replace $g^i(y)$ with

$$g^i(y) \simeq g^i_{k,M}(x) = G_i^T \psi(x) = \psi^T(x) G_i, \tag{20}$$

where U_i and G_i are unknown vectors of dimension $k(M + 1)$. Then, substituting these approximations and using the operational matrix of dual in (19) yields:

$$U_n = F_n + \frac{t_n}{2n} \left[\kappa_{n,0} D G_0 + \kappa_{n,n} D G_n + 2 \sum_{i=1}^{n-1} \kappa_{n,i} D G_i \right], \tag{21}$$

which forms a system of $k(M + 1)$ linear algebraic equations in terms of $2k(M + 1)$ unknowns. In order to obtain a uniquely solvable system, we need $k(M+1)$ additional equations. For this purpose consider $k(M + 1)$ collocation points defined by

$$x_{i,j} = \frac{h}{2}(x_j + 2i - 1), \quad i = 1,\ 2, \ldots, k, \quad j = 0,\ 1, \ldots, M,$$

where x_j, $j = 0,\ 1, \ldots, M$ are the roots of Legendre polynomial of degree $M + 1$. Collocating the equation $g(x, t_n, U_n^T \psi(x)) = G_n^T \psi(x)$ in $x_{i,j}$, we obtain

$$g(x_{i,j}, t_n, U_n^T \psi(x_{i,j})) - G_n^T \psi(x_{i,j}) = 0, \quad \text{for } i = 1,\ 2, \ldots, k, \quad j = 0,\ 1, \ldots, M, \tag{22}$$

which is a system of $k(M+1)$ nonlinear equations in terms of the unknown elements of the vectors U_n and G_n. Finally, systems (21) and (22) together form a system of $2k(M + 1)$ equations and can be solved in terms of U_n and G_n using the Newton's iterative method. In the case $t = 0$, we have $U_0 = F_0$, and G_0 is obtained using the approximation of the function $g(x, 0, U_0^T \psi(x))$ (which is a known function) by the hybrid Legendre functions.

5. Numerical examples

In this section, the results of two numerical experiments are presented to validate accuracy, applicability and convergence of the proposed methods. In order to investigate the error of the method we introduce the following notations. The error norm is denoted by

$$e_n(x) = |u_n(x) - \tilde{u}_n(x)|,$$
$$E_{k,M,N}(t_n) = \|e_n(x)\|_2,$$

where $u_n(x)$ and $\tilde{u}_n(x)$ are the exact solution and the computed solution by the presented method at $t = t_n$ with selected k, M and N, respectively. For the convergence order, with respect to h, we use the estimate:

$$\rho_k(t_n) = \log_2 \left(E_{k,M,N}/E_{2k,M,N}\right);$$

and for the convergence order, with respect to τ, we write

$$\varrho_N(t_n) = \log_2 \left(E_{k,M,N}/E_{k,M,2N}\right).$$

When using different meshes in space (time), the stepsize h (resp. τ) of each subsequent mesh is twice smaller.

Example 1: Consider the following linear mixed Volterra-Fredholm integral equation as discussed in [5]

$$u(x,t) = f(x,t) + \int_0^t \int_0^2 K(x,t,y,z)u(y,z)dydz, \quad 0 \le t \le 1, \tag{23}$$

where

$$f(x,t) = \mathrm{e}^{-t}\left(\cos(x) + t\cos(x) + \frac{1}{2}t\cos(x-2)\sin(2)\right),$$
$$K(x,t,y,z) = -\cos(x-y)\mathrm{e}^{-(t-z)},$$

with the exact solution $u(x,t) = \mathrm{e}^{-t}\cos(x)$. After multiplying the exact solution by the kernel K we observe that the integrand function on the right-hand side of (23) does not depend on z. Therefore, the outer integral can be computed exactly and the final error of the numerical solution does not depend on τ. This is why in our tests we only check the convergence of the method, as $h \to 0$. We have applied the described numerical method with $M = 3$ and $M = 6$. In both cases, we have taken $N = 100$ and used three different meshes in space, with $k = 2,\ 4,\ 8$. The numerical results are given in Tables 1–2. They present 4-th order convergence in the case $M = 3$ and 7-th order convergence in the case $M = 6$.

Example 2: Consider the following nonlinear mixed Volterra-Fredholm integral equation, which arises in the mathematical modeling of the development of an epidemic [1, 3]:

$$u(x,t) = f(x,t) + \int_0^t \int_0^1 K(x,t,y,z)(1 - \mathrm{e}^{-u(y,z)})dydz \quad 0 \le t \le 1, \tag{24}$$

t	$E_{2,3,100}$	$E_{4,3,100}$	ρ_2	$E_{8,3,100}$	ρ_4
0.1	1.5659×10^{-4}	1.0029×10^{-5}	3.96	6.3032×10^{-7}	3.99
0.2	1.4168×10^{-4}	9.0747×10^{-6}	3.96	5.7034×10^{-7}	3.99
0.3	1.2820×10^{-4}	8.2111×10^{-6}	3.96	5.1606×10^{-7}	3.99
0.4	1.1600×10^{-4}	7.4297×10^{-6}	3.96	4.6695×10^{-7}	3.99
0.5	1.0496×10^{-4}	6.7227×10^{-6}	3.96	4.2252×10^{-7}	3.99
0.6	9.4976×10^{-5}	6.0829×10^{-6}	3.96	3.8231×10^{-7}	3.99
0.7	8.5938×10^{-5}	5.5040×10^{-6}	3.96	3.4593×10^{-7}	3.99
0.8	7.7760×10^{-5}	4.9803×10^{-6}	3.96	3.1301×10^{-7}	3.99
0.9	7.0360×10^{-5}	4.5063×10^{-6}	3.96	2.8322×10^{-7}	3.99
1.0	6.3664×10^{-5}	4.0775×10^{-6}	3.96	2.5627×10^{-7}	3.99

Table 1: Numerical results for Example 1

t	$E_{2,6,100}$	$E_{4,6,100}$	ρ_2	$E_{8,6,100}$	ρ_4
0.1	1.4847×10^{-8}	1.1527×10^{-10}	7.00	8.9936×10^{-13}	7.00
0.2	1.3434×10^{-8}	1.0430×10^{-10}	7.00	8.1378×10^{-13}	7.00
0.3	1.2156×10^{-8}	9.4374×10^{-11}	7.00	7.3634×10^{-13}	7.00
0.4	1.0999×10^{-8}	8.5393×10^{-11}	7.00	6.6626×10^{-13}	7.00
0.5	9.9528×10^{-9}	7.7267×10^{-11}	7.00	6.0286×10^{-13}	7.00
0.6	9.0056×10^{-9}	6.9914×10^{-11}	7.00	5.4549×10^{-13}	7.00
0.7	8.1486×10^{-9}	6.3261×10^{-11}	7.00	4.9358×10^{-13}	7.00
0.8	7.3732×10^{-9}	5.7241×10^{-11}	7.00	4.4661×10^{-13}	7.00
0.9	6.6715×10^{-9}	5.1794×10^{-11}	7.00	4.0411×10^{-13}	7.00
1.0	6.0366×10^{-9}	4.6865×10^{-11}	7.00	3.6565×10^{-13}	7.00

Table 2: Numerical results for Example 1

where

$$f(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right) + \frac{xt^2}{8(1+t)(1+t^2)},$$

$$K(x,t,y,z) = \frac{x(1-y^2)}{(1+t)(1+z^2)}.$$

Its exact solution is $u(x,t) = -\ln(1 + xt/(1+t^2))$. The results of the numerical experiments with this example are displayed in Tables 3–4. In Table 3, τ is kept constant, with $N = 1000$, and $M = 2$ (quadratic polynomials). Note that with such value of N resulting from the time discretization is negligible when compared with the final error, so we can again investigate the dependence of the error on h. The error norms on three different meshes ($k = 2$,$k = 4$, and $k = 8$) show that the discretization error depends on h as $O(h^3)$. Finally, we have investigated the depen-

t	$E_{2,2,1000}$	$E_{4,2,1000}$	ρ_2	$E_{8,2,1000}$	ρ_4
0.1	6.6527×10^{-7}	8.3258×10^{-8}	2.99	1.0430×10^{-8}	2.99
0.2	4.3442×10^{-6}	5.4526×10^{-7}	2.99	6.8263×10^{-8}	2.99
0.3	1.1613×10^{-5}	1.4632×10^{-6}	2.98	1.8332×10^{-7}	2.99
0.4	2.1268×10^{-5}	2.6906×10^{-6}	2.98	3.3741×10^{-7}	2.99
0.5	3.1480×10^{-5}	3.9972×10^{-6}	2.97	5.0173×10^{-7}	2.99
0.6	4.0666×10^{-5}	5.1791×10^{-6}	2.97	6.5060×10^{-7}	2.99
0.7	4.7867×10^{-5}	6.1096×10^{-6}	2.97	7.6794×10^{-7}	2.99
0.8	5.2746×10^{-5}	6.7421×10^{-6}	2.96	8.4777×10^{-7}	2.99
0.9	5.5411×10^{-5}	7.0882×10^{-6}	2.96	8.9148×10^{-7}	2.99
1.0	5.6206×10^{-5}	7.1916×10^{-6}	2.96	9.0452×10^{-7}	2.99

Table 3: Numerical results for Example 2

t	$E_{16,2,20}$	$E_{16,2,40}$	ϱ_{20}	$E_{16,2,80}$	ϱ_{40}
0.1	1.6164×10^{-6}	4.0346×10^{-7}	2.00	1.0083×10^{-7}	2.00
0.2	5.5511×10^{-6}	1.3861×10^{-6}	2.00	3.4654×10^{-7}	1.99
0.3	1.0347×10^{-5}	2.5844×10^{-6}	2.00	6.4635×10^{-7}	1.99
0.4	1.4797×10^{-5}	3.6966×10^{-6}	2.00	9.2489×10^{-7}	1.99
0.5	1.8199×10^{-5}	4.5474×10^{-6}	2.00	1.1383×10^{-6}	1.99
0.6	2.0349×10^{-5}	5.0854×10^{-6}	2.00	1.2736×10^{-6}	1.99
0.7	2.1372×10^{-5}	5.3413×10^{-6}	2.00	1.3384×10^{-6}	1.99
0.8	2.1534×10^{-5}	5.3825×10^{-6}	2.00	1.3494×10^{-6}	1.99
0.9	2.1122×10^{-5}	5.2798×10^{-6}	2.00	1.3242×10^{-6}	1.99
1.0	2.0371×10^{-5}	5.0923×10^{-6}	2.00	1.2777×10^{-6}	1.99

Table 4: Numerical results for Example 2

dence of the error on τ. With this purpose, we have used three different stepsizes in time corresponding to $N = 20, N = 40$ and $N = 80$, keeping the stepsize h fixed ($k = 16$). For such stepsizes, the error resulting from the space discretization is much smaller than the component depending on τ. In this case, the results displayed in Table 4 show clearly that the error behaves as τ^2.

6. Conclusion

A new method is proposed for the numerical solution of linear and nonlinear mixed Volterra-Fredholm integral equations. The numerical scheme combines the trapezoidal rule, for integration in time, and piecewise polynomial approximation, for space discretisation. The hybrid Legendre functions and the operational matrix of dual are applied to reduce the problem to an algebraic system of nonlinear equations, which is solved by the Newton method. The computational method was tested using a sample of numerical examples, including an equation arising in the modeling of spatio-temporal development of an epidemic (Example 2), which was formerly analysed by other authors [1, 3]. Our results for this example (see Tables 3–4) have the same degree of accuracy (6 digits) as the results presented in [3], obtained by means of a collocation scheme with Gaussian points, both in time and space. The numerical experiments suggest that the convergence order is $O(h^{M+1}) + O(\tau^2)$, which is in agreement with the known properties of methods based on piecewise polynomial collocation and trapezoidal rule. We leave as a future work the analysis of convergence.

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