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ON THE INTERPOLATION CONSTANTS OVER TRIANGULAR ELEMENTS

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Abstract: We propose a simple method to obtain sharp upper bounds for the interpolation error constants over the given triangular elements. These constants are important for analysis of interpolation error and especially for the error analysis in the Finite Element Method. In our method, interpolation constants are bounded by the product of the solution of corresponding finite dimensional eigenvalue problems and constant which is slightly larger than one. Guaranteed upper bounds for these constants are obtained via the numerical verification method. Furthermore, we introduce remarkable formulas for the upper bounds of these constants.

Keywords: interpolation error constant, numerical verification method, Finite Element Method

MSC: 65D05, 65N15, 65D30

1. Introduction

The analysis of interpolation error is important in a lot of applications such as the approximate theory and the error estimation for the solution of Finite Element Method. In order to estimate the interpolation errors, we have to obtain the upper bounds of the constants which appear in some kinds of norm inequalities. These are called interpolation error constants.

Let T be given triangle in $\mathbb{R}^2$ and define function spaces $V^{1,1}(T), V^{1,2}(T), V^2(T)$ as follows:

$$V^{1,1}(T) = \left\{ \varphi \in H^1(T) \;\middle|\; \int_T \varphi \, dxdy = 0 \right\},$$

$$V^{1,2}(T) = \left\{ \varphi \in H^1(T) \;\middle|\; \int_{\gamma_k} \varphi \, ds = 0, \quad \forall k = 1,2,3 \right\},$$

$$V^2(T) = \left\{ \varphi \in H^2(T) \;\middle|\; \varphi(p_k) = 0, \quad \forall k = 1,2,3 \right\},$$

where p_1, p_2, p_3 and $\gamma_1, \gamma_2, \gamma_3$ are vertices and edges of T, respectively. Under these settings, it is known that the following interpolation error constants $C_1(T)$, $C_2(T)$, $C_3(T)$ and $C_4(T)$ exist:

$$C_1(T) = \sup_{u \in V^{1,1}(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}}, \qquad C_2(T) = \sup_{u \in V^{1,2}(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}},$$

$$C_3(T) = \sup_{u \in V^{2}(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{|u|_{H^2(T)}}, \qquad C_4(T) = \sup_{u \in V^{2}(T) \backslash 0} \frac{\|\nabla u\|_{L^2(T)}}{|u|_{H^2(T)}}.$$

where $|\cdot|_{H^k(\Omega)}$ means H^k semi-norm defined later.

In this paper, we present a simple method to obtain explicit and sharp upper bounds for them. Furthermore, we obtained the following remarkable formulas for the upper bounds:

$$C_1(T) < K_1(T) = \sqrt{\frac{A^2+B^2+C^2}{28} - \frac{S^4}{A^2B^2C^2}},$$

$$C_2(T) < K_2(T) = \sqrt{\frac{A^2+B^2+C^2}{54} - \frac{S^4}{2A^2B^2C^2}},$$

$$C_3(T) < K_3(T) = \sqrt{\frac{A^2B^2+B^2C^2+C^2A^2}{83} - \frac{1}{24}\left(\frac{A^2B^2C^2}{A^2+B^2+C^2} + S^2\right)},$$

$$C_4(T) < K_4(T) = \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}\right)},$$

where A, B, C are the edge lengths of triangle T and S is the area of T.

As we will show in Section 5, the upper bounds obtained by these formulas are sharp enough for the practical applications. Moreover, $K_1(T) \dots K_4(T)$ are convenient for practical calculations since these formulas consists of just four arithmetic operations and the square root.

We have to note that, by our method, we can only prove these formulas for the "given" triangles. To prove the formulas for "any" triangle, we need some continuation techniques and the asymptotic analysis. More specifically, we first prove these formulas for finitely many specific triangles by slightly strict form, namely

$$C_j(T) < (1-\varepsilon)K_j(T)$$

for some small $\varepsilon > 0$ and then extend these results to general cases by the analytical evaluation and the asymptotic analysis. We indeed succeeded to prove it but we will show it in another paper because of the space limit.

2. Preceding works

In connection with the Finite Element Method, there is a plenty of works especially on the relation between $C_4(T)$ and the error estimates such as [4, 6, 3, 9, 10, 12, 19, 14, 24] for *a priori* error estimate and [4, 8, 14] for *a posteriori* error estimate.

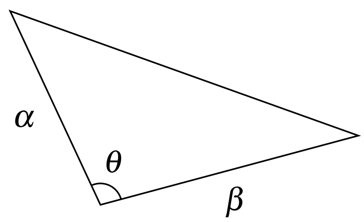


Figure 1: α, β and θ for triangle T.

On the explicit upper bound for $C_4(T)$, Arcangeli and Gout[2] obtained the following estimates:

$$C_4(T) \le \frac{3d(T)^2}{\rho(T)}$$

where $d(T)$ is a diameter of T and $\rho(T)$ is a diameter of the inscribed circle of T. They also obtained the upper bound for $C_3(T)$ as follows:

$$C_3(T) \le 3d(T)^2.$$

Meinguet and Descloux[17] improved their result and obtained

$$C_4(T) \le \frac{1.21d(T)^2}{\rho(T)}.$$

Natterer [20] showed that $C_4(T)$ is bounded in terms of $C_4(T_{0,1})$ where $T_{0,1}$ is a isosceles right triangle whose edge lengths are $1, 1$ and $\sqrt{2}$. Specifically, they showed

$$C_4(T) \le C_4(T_{0,1}) \cdot \frac{\alpha^2 + \beta^2 + \sqrt{\alpha^4 + 2\alpha^2\beta^2\cos 2\theta + \beta^4}}{\sqrt{2(\alpha^2 + \beta^2 - \sqrt{\alpha^4 + 2\alpha^2\beta^2\cos 2\theta + \beta^4})}}, \tag{1}$$

where α and β are the longest and second longest edge lengths and θ is an included angle (Fig. 1). In the same paper, they proved $C_4(T_{0,1}) \le 0.81$. Nakao and Yamamoto [19] proved that

$$C_4(T_{0,1}) \le 0.4939$$

by numerical verification method. Kikuchi and Liu [7] proved that $C_4(T_{0,1})$ is bounded by the maximum positive solution of transcendental equation for μ:

$$\frac{1}{\mu} + \tan\frac{1}{\mu} = 0$$

and showed

$$C_4(T_{0,1}) \le 0.49293.$$

Moreover, Liu and Kikuchi [14] proved that

$$C_4(T) \le C_4(T_{0,1}) \cdot \frac{1 + \cos\theta}{\sin\theta}\sqrt{\frac{\alpha^2 + \beta^2 + \sqrt{\alpha^4 + 2\alpha^2\beta^2\cos 2\theta + \beta^4}}{2}}. \tag{2}$$

Note that the estimation (2) is consistent with the maximum angle condition [3] whereas the estimation (1) is not. In fact, if we fix β and θ and let $\alpha \to 0$, the right-hand side of (1) diverges to infinity whereas the right-hand side of (2) remains bounded.

$C_1(T)$ is known as the Poincaré-Friedrichs constant and Payne and Weinberger obtained

$$C_1(T) \le \frac{d(T)}{\pi}.$$

This estimation is valid for any convex domain. For arbitrary triangle T, Laugesen and Siudeja [11] obtained

$$C_1(T) \le \frac{d(T)}{j_{1,1}} \tag{3}$$

where $j_{1,1} = 3.83170597\ldots$ denotes the first positive root of the Bessel function J_1.

On the other hand, Kikuchi and Liu [7] proved that

$$C_1(T_{0,1}) = \frac{1}{\pi}$$

and

$$C_1(T) \le C_1(T_{0,1})\sqrt{1 + |\cos\theta|}\ \max(\alpha, \beta). \tag{4}$$

There are only a few results for $C_2(T)$ itself. However, $C_2(T)$ is bounded by so called Babuška-Aziz constant whose existence is proved by Babuška and Aziz [3, Lemma 2.1]. From the upper bound for the Babuška-Aziz constant obtained by Liu and Kikuchi [14], we have

$$C_2(T) \le 0.34856\sqrt{1 + |\cos\theta|}\ \max(\alpha, \beta).$$

For the most triangles, our formulas $K_1(T) \ldots K_4(T)$ give better upper bounds than the preceding results. The exception is that (3) or (4) provides slightly lower value than $K_1(T)$ for some triangles.

There are some results about computing lower bounds of eigenvalues of elliptic operators such as [1, 5, 13, 15, 16, 21, 23] which can be applied to compute upper bounds of $C_1(T)$ or $C_2(T)$. Compared to these results, our method is only applicable to the triangular domain but has the advantage that the sharp upper bounds can be obtained by a simple implementation.

3. Definitions and preliminaries

For given triangle T, let $p_1(T), p_2(T), p_3(T)$ be vertices of T and $\gamma_1(T), \gamma_2(T), \gamma_3(T)$ be edges $p_2(T)p_3(T)$, $p_3(T)p_1(T)$, $p_1(T)p_2(T)$, respectively. Let $n(T)$ be the outer normal unit vector on ∂T, $\nu(T)$ be the direction vector which takes counterclockwise direction through ∂T and $ds(T)$ be the line element on ∂T. We omit "(T)" if there is no possibility of confusion. We use Cartesian coordinates (x, y) and use the usual notation for L^2 norm and define H^k semi-norm $|\cdot|_{H^k(T)}$ by $|u|^2_{H^k(\Omega)} =$

$\sum_{j=0}^{k} \binom{k}{j} \left\| \frac{\partial^k u}{\partial x^j \partial y^{k-j}} \right\|_{L^2(\Omega)}^2$. $T_{a,b}$ denotes triangle whose vertices are $(0,0)$, $(1,0)$ and (a,b). We use subscripts to indicate partial derivatives.

Let Q_α and Q_β denote the following polynomial spaces:

$$Q_\alpha = \left\{ a_1(x^2+y^2) + a_2 x + a_3 y + a_4 \;\middle|\; a_1, \ldots, a_4 \in \mathbb{R} \right\},$$
$$Q_\beta = \left\{ a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6 \;\middle|\; a_1, \ldots, a_6 \in \mathbb{R} \right\}.$$

Note that both Q_α and Q_β are invariant under constant shifts and rotations and thus they are independent of the choice of the coordinates. Let τ be the given triangle and we define two kinds of second order interpolation $\Pi_\tau^{(\alpha)}\varphi$ for $\varphi \in H^1(\tau)$ and $\Pi_\tau^{(\beta)}\varphi$ for $\varphi \in H^2(\tau)$ on triangle τ as follows:

$$\begin{cases} \Pi_\tau^{(\alpha)}\varphi \in Q_\alpha \\ \displaystyle\int_{\gamma_k} \Pi_\tau^{(\alpha)}\varphi \, ds = \int_{\gamma_k} \varphi \, ds, \qquad k = 1,2,3, \\ \displaystyle\iint_\tau \Pi_\tau^{(\alpha)}\varphi \, dxdy = \iint_\tau \varphi \, dxdy, \end{cases}$$

$$\begin{cases} \Pi_\tau^{(\beta)}\varphi \in Q_\beta \\ \Pi_\tau^{(\beta)}\varphi(p_k) = \varphi(p_k), \qquad k = 1,2,3, \\ \displaystyle\int_{\gamma_k} \nabla\Pi_\tau^{(\beta)}\varphi \cdot n \, ds = \int_{\gamma_k} \nabla\varphi \cdot n \, ds, \qquad k = 1,2,3. \end{cases}$$

In the rest of this section, we prepare some preliminary lemmas.

Lemma 1. *If $\varphi \in V^2(\tau)$ satisfies*

$$\int_{\gamma_k} \nabla\varphi \cdot n \, ds = 0, \quad k = 1,2,3,$$

then

$$\varphi_x,\ \varphi_y \in V^{1,2}(\tau)$$

holds.

Proof. From $\varphi(p_1) = \varphi(p_2) = \varphi(p_3) = 0$, we have

$$\int_{\gamma_k} \nabla\varphi \cdot \nu \, ds = 0, \qquad k = 1,2,3.$$

Then, together with the assumption,

$$\int_{\gamma_k} \nabla\varphi \cdot w \, ds = 0, \qquad k = 1,2,3,$$

holds for any fixed vector w, which proves the lemma. □

On the interpolations $\Pi_\tau^{(\alpha)}$ and $\Pi_\tau^{(\beta)}$, the following orthogonal properties hold:

Lemma 2. *For* $\varphi \in H^1(\tau)$,

$$\|\nabla \Pi_\tau^{(\alpha)}\varphi\|_{L^2(\tau)}^2 + \|\nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi)\|_{L^2(\tau)}^2 = \|\nabla\varphi\|_{L^2(\tau)}^2.$$

Lemma 3. *For* $\varphi \in H^2(\tau)$,

$$|\Pi_\tau^{(\beta)}\varphi|_{H^2(\tau)}^2 + |\varphi - \Pi_\tau^{(\beta)}\varphi|_{H^2(\tau)}^2 = |\varphi|_{H^2(\tau)}^2.$$

Proof of Lemma 2. Since $\Pi_\tau^{(\alpha)}\varphi$ does not depend on the choice of the coordinates, we consider the x-axis to be aligned with the edge γ_3 and take $p_1 = (0,0)$, $p_2 = (h,0)$, $p_3 = (ah, bh)$ and

$$\Pi_\tau^{(\alpha)}\varphi = a_1(x^2+y^2) + a_2 x + a_3 y + a_4.$$

Then, the divergence theorem yields

$$\begin{aligned}
&\|\nabla\varphi\|_{L^2(\tau)}^2 - \|\nabla\Pi_\tau^{(\alpha)}\varphi\|_{L^2(\tau)}^2 - \|\nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi)\|_{L^2(\tau)}^2 \\
&\quad = 2\iint_\tau \nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi)\cdot\nabla\Pi_\tau^{(\alpha)}\varphi\,dxdy \\
&\quad = 2\iint_\tau \operatorname{div}\Big((\varphi - \Pi_\tau^{(\alpha)}\varphi)\nabla\Pi_\tau^{(\alpha)}\varphi\Big)dxdy - 2\iint_\tau (\varphi - \Pi_\tau^{(\alpha)}\varphi)\Delta\Pi_\tau^{(\alpha)}\varphi\,dxdy \\
&\quad = 2\oint_{\partial\tau}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\nabla\Pi_\tau^{(\alpha)}\varphi\cdot n\,ds - 8a_1\iint_\tau(\varphi - \Pi_\tau^{(\alpha)}\varphi)\,dxdy \\
&\quad = 2\oint_{\partial\tau}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\begin{pmatrix}2a_1x + a_2\\ 2a_1y + a_3\end{pmatrix}\cdot n\,ds \\
&\quad = 4a_1\oint_{\partial\tau}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\begin{pmatrix}x\\ y\end{pmatrix}\cdot n\,ds \\
&\quad = 4a_1\int_{\gamma_1}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\begin{pmatrix}x-h\\ y\end{pmatrix}\cdot n\,ds + 4a_1\int_{\gamma_2}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\begin{pmatrix}x-ah\\ y-bh\end{pmatrix}\cdot n\,ds \\
&\qquad\qquad + 4a_1\int_{\gamma_3}(\varphi - \Pi_\tau^{(\alpha)}\varphi)\begin{pmatrix}x\\ y\end{pmatrix}\cdot n\,ds \\
&\quad = 4a_1\int_{\gamma_1}\sqrt{(x-h)^2+y^2}\,(\varphi - \Pi_\tau^{(\alpha)}\varphi)\,\nu\cdot n\,ds \\
&\qquad + 4a_1\int_{\gamma_2}\sqrt{(x-ah)^2+(y-bh)^2}\,(\varphi - \Pi_\tau^{(\alpha)}\varphi)\,\nu\cdot n\,ds \\
&\qquad\qquad + 4a_1\int_{\gamma_3}\sqrt{x^2+y^2}\,(\varphi - \Pi_\tau^{(\alpha)}\varphi)\,\nu\cdot n\,ds = 0
\end{aligned}$$

□

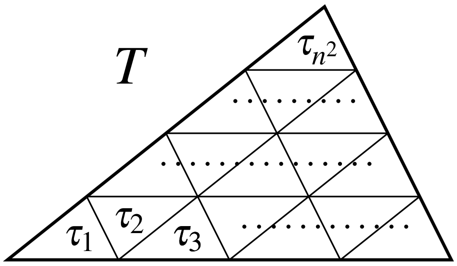


Figure 2: Divide T into n^2 congruent small triangles.

Proof of Lemma 3. Same as previous lemma, we take $p_1 = (0,0),\ p_2 = (h,0),\ p_3 = (ah, bh)$ and

$$\Pi_\tau^{(\beta)}\varphi = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6.$$

Then, the divergence theorem yields

$$\begin{aligned}
&|\varphi|^2_{H^2(\tau)} - |\Pi_\tau^{(\beta)}\varphi|^2_{H^2(\tau)} - |\varphi - \Pi_\tau^{(\beta)}\varphi|^2_{H^2(\tau)} \\
&\quad = 2\iint_\tau \Big((\varphi - \Pi_\tau^{(\beta)}\varphi)_{xx}(\Pi_\tau^{(\beta)}\varphi)_{xx} + 2(\varphi - \Pi_\tau^{(\beta)}\varphi)_{xy}(\Pi_\tau^{(\beta)}\varphi)_{xy} \\
&\qquad\qquad + (\varphi - \Pi_\tau^{(\beta)}\varphi)_{yy}(\Pi_\tau^{(\beta)}\varphi)_{yy}\Big)\,dxdy \\
&\quad = 2\iint_\tau \operatorname{div}\begin{pmatrix}\nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\nabla(\Pi_\tau^{(\beta)}\varphi)_x \\ \nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\nabla(\Pi_\tau^{(\beta)}\varphi)_y\end{pmatrix} dxdy \\
&\quad = 2\oint_{\partial\tau} \begin{pmatrix}\nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\nabla(\Pi_\tau^{(\beta)}\varphi)_x \\ \nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\nabla(\Pi_\tau^{(\beta)}\varphi)_y\end{pmatrix} \cdot n\,ds \\
&\quad = 2\oint_{\partial\tau} \nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\nabla(\nabla\Pi_\tau^{(\beta)}\varphi\cdot n)\,ds \\
&\quad = 2\oint_{\partial\tau} \nabla(\varphi - \Pi_\tau^{(\beta)}\varphi)\cdot\begin{pmatrix}2a_1 & a_2 \\ a_2 & 2a_3\end{pmatrix} n\,ds.
\end{aligned}$$

Here, Lemma 1 yields

$$\int_{\gamma_k} (\varphi - \Pi_\tau^{(\beta)}\varphi)_x\,ds = \int_{\gamma_k} (\varphi - \Pi_\tau^{(\beta)}\varphi)_y\,ds = 0, \qquad k = 1,2,3,$$

which leads us to the conclusion. □

4. Our method to bound the constants

We divide triangle T into n^2 congruent small triangles $\tau_1, \ldots, \tau_{n^2}$ (Fig. 2). We assume that each τ_k is open set, namely, does not contain its boundary, and define

$$T' = \bigcup_{k=1}^{n^2} \tau_k.$$

Then we define $\Pi^{(\alpha)}u$ for $u \in H^1(T)$ and $\Pi^{(\beta)}u$ for $u \in H^2(T)$ as follows:

$$\Pi^{(\alpha)}u|_{\tau_k} = \Pi^{(\alpha)}_{\tau_k}u, \qquad \Pi^{(\beta)}u|_{\tau_k} = \Pi^{(\beta)}_{\tau_k}u.$$

Note that $\Pi^{(\alpha)}u$ and $\Pi^{(\beta)}u$ are not always continuous on T.

By solving finite dimensional generalized eigenvalue problems, we can obtain following constants:

$$C_1^{(n)}(T) = \sup_{u\in V^{1,1}(T)\backslash 0} \frac{\|\Pi^{(\alpha)}u\|_{L^2(T')}}{\|\nabla\Pi^{(\alpha)}u\|_{L^2(T')}}, \qquad C_2^{(n)}(T) = \sup_{u\in V^{1,2}(T)\backslash 0} \frac{\|\Pi^{(\alpha)}u\|_{L^2(T')}}{\|\nabla\Pi^{(\alpha)}u\|_{L^2(T')}},$$
$$C_3^{(n)}(T) = \sup_{u\in V^{2}(T)\backslash 0} \frac{\|\Pi^{(\beta)}u\|_{L^2(T')}}{|\Pi^{(\beta)}u|_{H^2(T')}}, \qquad C_4^{(n)}(T) = \sup_{u\in V^{2}(T)\backslash 0} \frac{\|\nabla\Pi^{(\beta)}u\|_{L^2(T')}}{|\Pi^{(\beta)}u|_{H^2(T')}}.$$

With respect to these constants, we have the following theorem:

Theorem 1.

$$C_1(T) \le \sqrt{\frac{n^2}{n^2-1}}\, C_1^{(n)}(T), \qquad C_2(T) \le \sqrt{\frac{n^2}{n^2-1}}\, C_2^{(n)}(T),$$
$$C_3(T) \le \sqrt{\frac{n^4}{n^4-1}}\, C_3^{(n)}(T), \qquad C_4(T) \le \sqrt{\frac{n^2}{n^2-1}}\, C_4^{(n)}(T),$$
$$C_4(T) \le \sqrt{C_4^{(n)}(T)^2 + \frac{C_2(T)^2}{n^2}},$$

Proof. We first note that the scaling properties $C_j(\tau_k) = C_j(T)/n$ for $j = 1,2,4$ and $C_3(\tau_k) = C_3(T)/n^2$ hold. This property can be easily shown by change of variables.

From Lemma 2, for $u \in V^{1,j}(T)$, $j = 1,2$, we have

$$\begin{aligned}
\|u\|_{L^2(T)} &\le \|\Pi^{(\alpha)}u\|_{L^2(T')} + \|u - \Pi^{(\alpha)}u\|_{L^2(T')} \\
&= \|\Pi^{(\alpha)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|u - \Pi^{(\alpha)}_{\tau_k}u\|^2_{L^2(\tau_k)}} \\
&\le C_j^{(n)}(T)\, \|\nabla\Pi^{(\alpha)}u\|_{L^2(T')} + \frac{C_j(T)}{n}\sqrt{\sum_{k=1}^{n^2} \|\nabla(u - \Pi^{(\alpha)}_{\tau_k}u)\|^2_{L^2(\tau_k)}} \\
&\le \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2}\left(\|\nabla\Pi^{(\alpha)}_{\tau_k}u\|^2_{L^2(\tau_k)} + \|\nabla(u - \Pi^{(\alpha)}_{\tau_k}u)|^2_{L^2(\tau_k)}\right)} \\
&= \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2} \|\nabla u\|^2_{L^2(\tau_k)}} \\
&= \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}}\, \|\nabla u\|_{L^2(T)}.
\end{aligned}$$

Furthermore, from Lemma 3, for $u \in V^2(T)$,

$$
\begin{aligned}
\|u\|_{L^2(T)} &\le \|\Pi^{(\beta)}u\|_{L^2(T')} + \|u - \Pi^{(\beta)}u\|_{L^2(T')} \\
&= \|\Pi^{(\beta)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|u - \Pi^{(\beta)}_{\tau_k}u\|^2_{L^2(\tau_k)}} \\
&\le C_3^{(n)}(T)\ |\Pi^{(\beta)}u|_{H^2(T')} + \frac{C_3(T)}{n^2}\sqrt{\sum_{k=1}^{n^2} |u - \Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)}} \\
&\le \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}}\sqrt{\sum_{k=1}^{n^2}\left(|\Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)} + |u - \Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)}\right)} \\
&= \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}}\sqrt{\sum_{k=1}^{n^2} |u|^2_{H^2(\tau_k)}} \\
&= \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}}\ |u|_{H^2(T)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla u\|_{L^2(T)} &\le \|\nabla\Pi^{(\beta)}u\|_{L^2(T')} + \|\nabla(u - \Pi^{(\beta)}u)\|_{L^2(T')} \\
&= \|\nabla\Pi^{(\beta)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|\nabla(u - \Pi^{(\beta)}_{\tau_k}u)\|^2_{L^2(\tau_k)}} \\
&\le C_4^{(n)}(T)\ |\Pi^{(\beta)}u|_{H^2(T')} + \frac{C_4(T)}{n}\sqrt{\sum_{k=1}^{n^2} |u - \Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)}} \\
&\le \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2}\left(|\Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)} + |u - \Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)}\right)} \\
&= \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2} |u|^2_{H^2(\tau_k)}} \\
&= \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\ |u|_{H^2(T)}
\end{aligned}
$$

hold. Using Lemma 1, we can also evaluate $\|\nabla(u - \Pi^{(\beta)}u)\|_{L^2(T')}$ in the first line of the previous expression by

$$
\begin{aligned}
\|\nabla(u-\Pi^{(\beta)}u)\|_{L^2(T')} &= \sqrt{\sum_{k=1}^{n^2}\left(\|(u-\Pi^{(\beta)}_{\tau_k}u)_x\|^2_{L^2(\tau_k)}+\|(u-\Pi^{(\beta)}_{\tau_k}u)_y\|^2_{L^2(\tau_k)}\right)}\\
&\le \frac{C_2(T)}{n}\sqrt{\sum_{k=1}^{n^2}\left(\|\nabla(u-\Pi^{(\beta)}_{\tau_k}u)_x\|^2_{L^2(\tau_k)}+\|\nabla(u-\Pi^{(\beta)}_{\tau_k}u)_y\|^2_{L^2(\tau_k)}\right)}\\
&= \frac{C_2(T)}{n}\sqrt{\sum_{k=1}^{n^2}|u-\Pi^{(\beta)}_{\tau_k}u|^2_{H^2(\tau_k)}}.
\end{aligned}
$$

From above evaluations, we have the following:

$$
\begin{aligned}
C_1(T) &\le \sqrt{C_1^{(n)}(T)^2+\frac{C_1(T)^2}{n^2}}, & C_2(T) &\le \sqrt{C_2^{(n)}(T)^2+\frac{C_2(T)^2}{n^2}},\\
C_3(T) &\le \sqrt{C_3^{(n)}(T)^2+\frac{C_3(T)^2}{n^4}}, & C_4(T) &\le \sqrt{C_4^{(n)}(T)^2+\frac{C_4(T)^2}{n^2}},\\
C_4(T) &\le \sqrt{C_4^{(n)}(T)^2+\frac{C_2(T)^2}{n^2}},
\end{aligned}
$$

which leads us to the conclusion. □

This result shows that we can bound the constants $C_1(T)\dots C_4(T)$ by means of $C_1^{(n)}(T)\dots C_4^{(n)}(T)$. We can compute $C_1^{(n)}(T)\dots C_4^{(n)}(T)$ numerically and also obtain guaranteed results via the numerical verification method.

5. Numerical results

In this section, we show the values of the upper bounds for $C_1(T)\dots C_4(T)$ obtained by Theorem 1, that of $K_1(T)\dots K_4(T)$ in Section 1 and that of $C_1(T)\dots C_4(T)$ themselves. We can calculate $C_1^{(n)}(T)\dots C_4^{(n)}(T)$ via the numerical verification method with interval arithmetic using INTLAB, the MATLAB toolbox for the reliable computing [18, 22]. Let $\overline{C}_1^{(n)}(T)\dots\overline{C}_4^{(n)}(T)$ be the upper endpoints of the calculated intervals by INTLAB, then from Theorem 1, the upper bounds for $C_1(T)\dots C_4(T)$ are obtained as follows:

$$
\begin{aligned}
\overline{\overline{C}}_1^{(n)}(T) &= \sqrt{\frac{n^2}{n^2-1}}\,\overline{C}_1^{(n)}(T), & \overline{\overline{C}}_2^{(n)}(T) &= \sqrt{\frac{n^2}{n^2-1}}\,\overline{C}_2^{(n)}(T),\\
\overline{\overline{C}}_3^{(n)}(T) &= \sqrt{\frac{n^4}{n^4-1}}\,\overline{C}_3^{(n)}(T), & \overline{\overline{C}}_4^{(n)}(T) &= \sqrt{\frac{n^2}{n^2-1}}\,\overline{C}_4^{(n)}(T),\\
\overline{\overline{C}}\,{}'^{(n)}_4(T) &= \sqrt{\overline{C}_4^{(n)}(T)^2+\frac{\overline{\overline{C}}_2^{(n)}(T)^2}{n^2}}.
\end{aligned}
$$

As for $C_1(T)\dots C_4(T)$ themselves, we cannot determine their values analytically. Therefore, we first compute the following values for $n \le 10$:

$$
\widetilde{C}_1^{(n)}(T) = \sup_{u \in V^{1,1}(T)\cap\mathcal{P}_n\backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}}, \qquad \widetilde{C}_2^{(n)}(T) = \sup_{u \in V^{1,2}(T)\cap\mathcal{P}_n\backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}},
$$
$$
\widetilde{C}_3^{(n)}(T) = \sup_{u \in V^{2}(T)\cap\mathcal{P}_n\backslash 0} \frac{\|u\|_{L^2(T)}}{|u|_{H^2(T)}}, \qquad \widetilde{C}_4^{(n)}(T) = \sup_{u \in V^{2}(T)\cap\mathcal{P}_n\backslash 0} \frac{\|\nabla u\|_{L^2(T)}}{|u|_{H^2(T)}},
$$

where $\mathcal{P}_n$ denote the space of polynomials with degree less than or equal to n, then apply the repeated Aitken extrapolation to obtain more accurate approximations $\widetilde{C}_1(T)\dots\widetilde{C}_4(T)$.

In the following tables, all numerical results are rounded up to seven decimal places. Note that $T_{a,b},\ 0 \le a \le 0.5,\ 0 < b \le 1$ provides all shapes of triangles and, due to the scaling property, the relative error between the upper bounds and the optimal values depends only on the shape of the triangle.

The numerical results show that the sharp and explicit upper bounds are obtained by our method and the formulas introduced in Section 1. We also checked that

$$
\begin{aligned}
&\overline{\overline{C}}_j^{(20)}(T_{a,b}) < K_j(T_{a,b}), \qquad j = 1,2,3,\\
&\overline{\overline{C}}{}'^{(20)}_4(T_{a,b}) < K_4(T_{a,b}),
\end{aligned}
$$

holds for every triangles with $(a,b) = (k/100, l/100),\ 0 \le k \le 50,\ 1 \le l \le 100$.

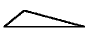
T	Shape	$K_1(T)$	$\overline{\overline{C}}_1^{(10)}(T)$	$\overline{\overline{C}}_1^{(20)}(T)$	$\widetilde{C}_1(T)$
$T_{0,1}$	 Isosceles right triangle	0.3340766	0.3212290	0.3190436	0.3183099
$T_{0,1/2}$		0.2771024	0.2740807	0.2723761	0.2718064
$T_{0,1/5}$		0.2681080	0.2648395	0.2632425	0.2627047
$T_{0,1/10}$		0.2674398	0.2635352	0.2619488	0.2614141
$T_{1/4,1}$		0.3030136	0.2911752	0.2893022	0.2886729
$T_{1/4,1/2}$		0.2459843	0.2436090	0.2420943	0.2415907
$T_{1/4,1/5}$		0.2434617	0.2329771	0.2312917	0.2307200
$T_{1/4,1/10}$		0.2420732	0.2310303	0.2292291	0.2285833
$T_{1/2,\sqrt{3}/2}$	 Equilateral triangle	0.2683033	0.2408094	0.2392551	0.2387325
$T_{1/2,1/2}$	 Isosceles right triangle	0.2362278	0.2271432	0.2255927	0.2250791
$T_{1/2,1/5}$		0.2350309	0.2150884	0.2129926	0.2122547
$T_{1/2,1/10}$		0.2327945	0.2124695	0.2100807	0.2091564

Table 1: Calculation results for $C_1(T)$.

6. Circumradius and $C_4(T)$

In Section 1, we claimed that the following estimate holds for the interpolation constant $C_4(T)$:

$$C_4(T) < K_4(T) = \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right)},$$

T	Shape	$K_2(T)$	$\overline{\overline{C}}_2^{(10)}(T)$	$\overline{\overline{C}}_2^{(20)}(T)$	$\widetilde{C}_2(T)$
$T_{0,1}$	Isosceles right triangle	0.2417625	0.2396039	0.2381772	0.2377024
$T_{0,1/2}$		0.2001158	0.1998408	0.1985657	0.1981418
$T_{0,1/5}$		0.1931751	0.1916921	0.1904436	0.1900288
$T_{0,1/10}$		0.1926085	0.1906412	0.1893972	0.1889838
$T_{1/4,1}$		0.2197865	0.2177021	0.2164124	0.2159829
$T_{1/4,1/2}$		0.1779313	0.1782025	0.1770818	0.1767091
$T_{1/4,1/5}$		0.1753980	0.1720157	0.1709011	0.1705287
$T_{1/4,1/10}$		0.1743207	0.1711858	0.1700506	0.1696660
$T_{1/2,\sqrt{3}/2}$	Equilateral triangle	0.1948780	0.1906371	0.1895418	0.1891770
$T_{1/2,1/2}$	Isosceles right triangle	0.1709519	0.1694255	0.1684167	0.1680810
$T_{1/2,1/5}$		0.1693067	0.1645693	0.1635627	0.1632276
$T_{1/2,1/10}$		0.1676363	0.1638830	0.1628606	0.1625187

Table 2: Calculation results for $C_2(T)$.

T	Shape	$K_3(T)$	$\overline{\overline{C}}_3^{(10)}(T)$	$\overline{\overline{C}}_3^{(20)}(T)$	$\widetilde{C}_3(T)$
$T_{0,1}$	Isosceles right triangle	0.1702674	0.1684446	0.1675538	0.1672540
$T_{0,1/2}$		0.1184266	0.1180690	0.1175455	0.1173699
$T_{0,1/5}$		0.1107396	0.1096648	0.1092458	0.1091056
$T_{0,1/10}$		0.1099925	0.1087203	0.1083189	0.1081843
$T_{1/4,1}$		0.1487598	0.1464850	0.1458512	0.1456392
$T_{1/4,1/2}$		0.0950296	0.0946780	0.0942616	0.0941222
$T_{1/4,1/5}$		0.0855113	0.0849795	0.0844707	0.0842867
$T_{1/4,1/10}$		0.0843545	0.0837111	0.0831606	0.0829448
$T_{1/2,\sqrt{3}/2}$	Equilateral triangle	0.1201799	0.1177043	0.1172419	0.1170872
$T_{1/2,1/2}$	Isosceles right triangle	0.0851337	0.0842223	0.0837769	0.0836270
$T_{1/2,1/5}$		0.0732579	0.0727068	0.0719786	0.0716964
$T_{1/2,1/10}$		0.0715702	0.0710650	0.0702398	0.0698864

Table 3: Calculation results for $C_3(T)$.

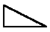
T	Shape	$K_4(T)$	$\overline{\overline{C}}_4^{(10)}(T)$	$\overline{\overline{C}}_4^{\prime(10)}(T)$	$\overline{\overline{C}}_4^{\prime(20)}(T)$	$\widetilde{C}_4(T)$
$T_{0,1}$	 Isosceles right triangle	0.4915961	0.4912760	0.4894003	0.4888906	0.4887225
$T_{0,1/2}$		0.3958115	0.3827571	0.3813624	0.3809004	0.3807482
$T_{0,1/5}$		0.3697886	0.3384254	0.3372742	0.3367584	0.3365883
$T_{0,1/10}$		0.3662945	0.3297106	0.3286114	0.3280661	0.3278854
$T_{1/4,1}$		0.4063828	0.3983769	0.3969774	0.3964682	0.3963006
$T_{1/4,1/2}$		0.3393941	0.3273684	0.3262146	0.3257826	0.3256403
$T_{1/4,1/5}$		0.5516444	0.5415574	0.5391173	0.5389133	0.5388452
$T_{1/4,1/10}$		0.9871946	0.9796800	0.9749196	0.9748225	0.9747889
$T_{1/2,\sqrt{3}/2}$	 Equilateral triangle	0.3476109	0.3200270	0.3189930	0.3185477	0.3184013
$T_{1/2,1/2}$	 Isosceles right triangle	0.3476109	0.3473846	0.3460583	0.3456979	0.3455790
$T_{1/2,1/5}$		0.6761400	0.6663349	0.6631990	0.6630533	0.6630043
$T_{1/2,1/10}$		1.2786662	1.2752049	1.2689187	1.2688525	1.2688286

Table 4: Calculation results for $C_4(T)$.

where A, B, C are the edge lengths of triangle T and S is the area of T. Since the circumradius of T is given by

$$R(T) = \frac{ABC}{4S},$$

we have the estimation

$$C_4(T) < R(T).$$

This fact is full of interesting suggestions for the error analysis in the Finite Element Method. See [9, 10] for the details.

7. Conclusion

We present a simple method to obtain sharp upper bounds for the interpolation error constants over the given triangular elements. These constants are important for analysis of interpolation error and especially for the error analysis in the Finite Element Method. Guaranteed upper bounds for these constants are obtained via the numerical verification method. Furthermore, we introduce remarkable formulas for the upper bounds of these constants. By the method explained in this paper, we can only prove these formulas for the given triangles. However, using some continuation techniques and asymptotic analysis, we are able to extend our results to the general cases. We will show the general proof in a forthcoming publication.

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