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ON CONTINUOUS AND DISCRETE MAXIMUM/MINIMUM PRINCIPLES FOR REACTION-DIFFUSION PROBLEMS WITH THE NEUMANN BOUNDARY CONDITION

István Faragó[1], Sergey Korotov[2], Tamás Szabó[3]

[1] Department of Applied Analysis and Computational Mathematics
Eötvös Loránd University
H–1117, Budapest, Pázmány P. s. 1/c., Hungary
& MTA-ELTE NumNet Research Group
faragois@cs.elte.hu

[2] Department of Computing, Mathematics and Physics
Bergen University College
Inndalsveien 28, 5020 Bergen, Norway
sergey.korotov@hib.no

[3] CAE Engineering Kft.
H–1118, Budapest, Kelenhegyi ut 43, Hungary
klaymen1984@gmail.com

Abstract: In this work, we present and discuss continuous and discrete maximum/minimum principles for reaction-diffusion problems with the Neumann boundary condition solved by the finite element and finite difference methods.

Keywords: elliptic problem, Neumann boundary condition, maximum/minimum principle, discrete maximum/minimum principle

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1. Continuous maximum/minimum principles

Consider the following boundary-value problem of elliptic type: Find a function $u \in C^2(\overline{\Omega})$ such that

$$-\Delta u + cu = f \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbf{R}^d$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$, n is the unit outward normal to $\partial\Omega$, and the reactive coefficient $c(x) \geq 0$ for all $x \in \overline{\Omega}$. The boundary condition in (1) is commonly called the *the Neumann boundary condition.*

The additional assumptions on the data of the problem will be given in appropriate places of the paper later on.

First, we prove the continuous *maximum/minimum principles* for problem (1) in the following form.

Theorem 1. *Assume that in* (1) *the functions* $c, f \in C(\overline{\Omega}), g \in C(\partial\Omega)$, *and* $c(x) \geq c_\star > 0$ *for all* $x \in \overline{\Omega}$, *where* $c_\star$ *is a positive constant. Let*

$$g(s) \leq -g_\star < 0 \quad \textit{for} \quad \textit{all} \quad s \in \partial\Omega, \tag{2}$$

where $g_\star$ *is a positive constant. Then the following a priori upper estimate (continuous maximum principle) for the classical solution of problem* (1) *is valid for any* $x \in \overline{\Omega}$:

$$u(x) \leq \max_{\bar{x} \in \overline{\Omega}} \frac{f(\bar{x})}{c(\bar{x})}. \tag{3}$$

Now, let

$$g(s) \geq g_\star > 0 \quad \textit{for} \quad \textit{all} \quad s \in \partial\Omega, \tag{4}$$

where $g_\star$ *is a positive constant. Then the following a priori lower estimate (continuous minimum principle) for the classical solution of problem* (1) *is valid for any* $x \in \overline{\Omega}$:

$$u(x) \geq \min_{\bar{x} \in \overline{\Omega}} \frac{f(\bar{x})}{c(\bar{x})}. \tag{5}$$

Proof. First, we prove estimate (3). If u attains its maximum at some interior point $x_0 \in \Omega$, then all the first order partial derivatives $u_{x_i}(x_0) = 0$, and all the second order partial derivatives $u_{x_i x_i}(x_0) \leq 0$ for $i = 1, 2, \ldots, d$. Therefore, from the equation in (1) and the positivity of c we observe that $u(x_0) \leq f(x_0)/c(x_0)$. Now we claim that under the assumptions of the theorem the maximum of u cannot be attained on the boundary. Indeed, if u attains its maximum at some boundary point $s_0 \in \partial\Omega$, then, unavoidably, $0 \leq \frac{\partial u}{\partial n}(s_0) = g(s_0)$, which contradicts the assumption on g in (2).

Obviously, estimate (5) can be proved in a similar way under conditions in (4). $\square$

In what follows we will always assume that the following condition on the coefficient c holds

$$c(x) \geq c_\star > 0 \quad \text{for all} \quad x \in \overline{\Omega}, \tag{6}$$

where $c_\star$ is a positive constant.

The main goal of the paper is to construct suitable discrete analogues of (3) and (5), called the *discrete maximum/minimum principles*, and find practical conditions on the numerical schemes, namely the finite element method (FEM) and the finite difference method (FDM), providing their validity.

In most of available papers devoted to maximum principles for elliptic problems, see e.g. [9, 11] and references therein, continuous maximum (and minimum) principles usually take a form of implications involving certain sign-conditions. For example, for the equation from (1) combined with vanishing Dirichlet boundary condition, the maximum principle reads as follows:

$$f(x) \leq 0 \quad \text{in} \quad \overline{\Omega} \implies \max_{x \in \overline{\Omega}} u(x) \leq 0. \tag{7}$$

However, the implications with sign-conditions (like in (7)) have been recently generalized in [6, 7] to more general situations for problems with Dirichlet and Robin boundary condition. In this work we consider the case of Neumann problem and perform an analysis of some FE and FD schemes in the context of discrete maximum/minimum principles.

Remark 1. We mention that discrete maximum principles, besides their practical importance for imitating the nonnegativity of nonnegative physical quantities in numerical simulations, have been often used for proving stability and finding the rate of convergence of FD approximations, see e.g. [1, 2, 4], and for proving the convergence of FE approximations in the maximum norm, see e.g. [1, 5].

2. Discrete maximum principle

After discretization of problem (1) by many popular numerical techniques (e.g. by FEM and FDM) we arrive at the problem of solving $N \times N$ system of linear algebraic equations

$$\mathbf{Au} = \mathbf{F}, \tag{8}$$

where the vector of unknowns $\mathbf{u} = [u_1, \ldots, u_N]^T$ approximates the unknown solution u at certain selected points $B_1, \ldots, B_N$ of the solution domain Ω and its boundary $\partial\Omega$, and the vector $\mathbf{F} = [F_1, \ldots, F_N]^T$ approximates (in the sense depending on the nature of the actual numerical method used) the values $f(B_i)$ and $g(B_j)$.

In what follows, the entries of matrix $\mathbf{A}$ are denoted by a_{ij}, and all matrix and vector inequalities appearing in the text are always understood component-wise.

Definition 1. The square $N \times N$ matrix $\mathbf{M}$ is called *monotone* if

$$\mathbf{Mz} \geq 0 \quad \implies \quad \mathbf{z} \geq 0. \tag{9}$$

Equivalently, monotone matrices are characterized as follows (see e.g. [2, p. 119]).

Theorem 2. *The square $N \times N$ matrix $\mathbf{M}$ is monotone if and only if $\mathbf{M}$ is nonsingular and $\mathbf{M}^{-1} \geq 0$.*

Definition 2. The square $N \times N$ matrix $\mathbf{M}$ is called *M-matrix* if it is monotone and its entries $m_{ij} \leq 0$ for $i \neq j$.

It is obvious that for M-matrix $\mathbf{M} = (m_{ij})$, we have $m_{ii} > 0$ for all $i = 1, \ldots, N$.

Definition 3. The square $N \times N$ matrix $\mathbf{M}$ (with entries m_{ij}) is called *strictly diagonally dominant* (or SDD in short) if the values

$$\delta_i(\mathbf{M}) := |m_{ii}| - \sum_{j=1,\, j\neq i}^{N} |m_{ij}| > 0 \quad \text{for all} \quad i = 1, \ldots, N. \tag{10}$$

In [17] the following result is proved.

Theorem 3. *Let matrix* $\mathbf{A}$ *in system* (8) *be SDD and M-matrix. Then the following two-sided estimates for the entries of the solution* $\mathbf{u}$ *are valid*

$$\min_{j=1,\ldots,N} \frac{F_j}{\delta_j(\mathbf{A})} \leq u_i \leq \max_{j=1,\ldots,N} \frac{F_j}{\delta_j(\mathbf{A})}, \quad i = 1, \ldots, N. \tag{11}$$

As the estimates in (11) resemble the estimates in (3) and (5), it is natural to give the following definition.

Definition 4. We say that the solution $\mathbf{u}$ of system (8) with an SDD matrix $\mathbf{A}$ satisfies the *discrete maximum principle* corresponding to continuous maximum principle (3), if the upper estimate in (11) is valid, and, in addition, the following inequality

$$\max_{j=1,\ldots,N} \frac{F_j}{\delta_j(\mathbf{A})} \leq \max_{\bar{x}\in\overline{\Omega}} \frac{f(\bar{x})}{c(\bar{x})} \tag{12}$$

holds. Similarly, we say that the solution $\mathbf{u}$ of system (8) with an SDD matrix $\mathbf{A}$ satisfies the *discrete minimum principle* corresponding to continuous minimum principle (5), if the lower estimate in (11) is valid, and, in addition, the following inequality

$$\min_{j=1,\ldots,N} \frac{F_j}{\delta_j(\mathbf{A})} \geq \min_{\bar{x}\in\overline{\Omega}} \frac{f(\bar{x})}{c(\bar{x})} \tag{13}$$

holds.

Remark 2. In case of earlier versions of continuous and discrete maximum principles no estimates like (12) and (13) were, in fact, needed as one dealt there with various implications involving the sign-conditions only (cf. [4, 5, 13, 9]).

Remark 3. The validity of relations (12) and (13) is important for producing controllable numerical approximations, because under these conditions the approximate solutions (obtained by the FEM or the FDM for example) stay within the same bounds as the exact solutions and these bounds are a priori known from the continuous problem.

3. DMPs for the finite element (FE) schemes

The standard FE scheme is based on the so-called variational formulation of (1), which reads: Find $u \in H^1(\Omega)$ such that

$$a(u,v) = \mathcal{F}(v) \qquad \forall v \in H^1(\Omega), \tag{14}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega cuv dx\,,\ \mathcal{F}(v) = \int_\Omega fv dx + \int_{\partial\Omega} gv ds. \tag{15}$$

The existence and uniqueness of the weak solution u is provided by the Lax-Milgram lemma, the Friedrichs-type inequalities, and the assumption on c (6) (cf. [14, Chapt. 2]). (Actually, for the well-posedness in above, one can require less smoothness from the problem data, e.g. that $c \in L^\infty(\Omega), f \in L^2(\Omega), g \in L^2(\partial\Omega)$ only.)

Let $\mathcal{T}_h$ be a FE mesh of $\overline{\Omega}$ with interior nodes $B_1, \ldots, B_n$ lying in Ω and boundary nodes $B_{n+1}, \ldots, B_{n+n^\partial}$ lying on $\partial\Omega$. The elements of $\mathcal{T}_h$ will be denoted by the symbol T, possibly with subindices. Further, let the basis functions $\phi_1, \phi_2, \ldots, \phi_{n+n^\partial}$, associated with these nodes, have the following properties

$$\phi_i(B_j) = \delta_{ij},\ i,j = 1,\ldots,n+n^\partial,\ \phi_i \geq 0 \text{ in } \overline{\Omega},\ i = 1,\ldots,n+n^\partial,$$

$$\sum_{i=1}^{n+n^\partial} \phi_i \equiv 1 \text{ in } \overline{\Omega}, \tag{16}$$

where δ_{ij} is the Kronecker delta. Note that these properties are easily met for example for the lowest-order simplicial, block, and prismatic finite elements. The basis functions $\phi_1, \phi_2, \ldots, \phi_{n+n^\partial}$ are spanning a finite-dimensional subspace V_h of $H^1(\Omega)$.

The FE approximation of u is defined to be a function $u_h \in V_h$ such that

$$a(u_h, v_h) = \mathcal{F}(v_h) \qquad \forall v_h \in V_h, \tag{17}$$

whose existence and uniqueness are also provided by the Lax-Milgram lemma.

Remark 4. Algorithmically, $u_h = \sum_{i=1}^{n+n^\partial} u_i\phi_i$, where the coefficients u_i are the entries of the solution $\mathbf{u}$ of system (8) with $a_{ij} = a(\phi_i, \phi_j)$, $F_i = \mathcal{F}(\phi_i)$, and $N = n + n^\partial$. It is clear that, if properties (16) hold, the FE approximation u_h satisfies the bounds from (11) at each point of $\overline{\Omega}$ if all its nodal values u_i do satisfy them.

Lemma 1. *Assume that problem* (1) *under condition* (2) *is solved by the FEM with basis functions satisfying* (16)*. In addition, let the matrix* $\mathbf{A}$ *in the resulting system* $\mathbf{Au} = \mathbf{F}$ *be such that* $a_{ij} \leq 0$ *for* $i \neq j$*. Then* $\mathbf{A}$ *is SDD and estimates* (11) *are valid.*

Proof. From (15) and (2), it clearly follows that $a_{ii} = a(\phi_i, \phi_i) > 0$ for all $i = 1, \dots, n + n^\partial$. If $a_{ij} \le 0$ ($i \ne j$), we observe for $i = 1, \dots, n + n^\partial$ that

$$\delta_i(\mathbf{A}) = \sum_{j=1}^{n+n^\partial} a_{ij} = a(\phi_i, \sum_{j=1}^{n+n^\partial} \phi_j) = a(\phi_i, 1) = \int_\Omega c\phi_i dx > 0, \tag{18}$$

where the last (strict) inequality holds due to (2). Thus, the matrix $\mathbf{A}$ is always SDD for our type of problems. Moreover $\mathbf{A}$ is the Minkowski matrix, i.e. M-matrix (cf. [2, pp. 119–120]). Hence, estimates (11) are valid, due to Theorem 3, with $\delta_i(\mathbf{A})$ computed as in (18). □

The proofs of further estimates (12) and (13) strongly depend on the way we compute a_{ij} and F_j in real calculations. Below we consider in detail only the following representative case.

Theorem 4. *Assume that the coefficient c is a positive constant and that all entries a_{ij} and F_j in system* (8) *are computed exactly. Then estimates* (12) *and* (13)*, and therefore discrete maximum and minimum principles, corresponding to* (3) *and* (5)*, correspondingly, are valid provided $a_{ij} \le 0$ for $i \ne j$, and the relevant sign condition on g holds.*

Proof. Let us prove first (12) under condition $g(s) \le -g_\star < 0$. In view of (18), (15), (2), and the first mean value theorem for integration, we get

$$\frac{F_i}{\delta_i(\mathbf{A})} = \frac{\int_\Omega f\phi_i dx + \int_{\partial\Omega} g\phi_i ds}{\int_\Omega c\phi_i dx} \le \frac{\int_\Omega f\phi_i dx}{c\int_\Omega \phi_i dx} =$$

$$= \frac{f(x^\star)\int_\Omega \phi_i dx}{c\int_\Omega \phi_i dx} \le \max_{\xi\in\overline{\Omega}} \frac{f(\xi)}{c},$$

where $x^\star$ is some point from $\overline{\Omega}$ and i is an arbitrary index from the set $\{1, \dots, n+n^\partial\}$.

Similarly, we can prove (13) under condition $g(s) \ge g_\star > 0$. □

Remark 5. In fact, the entries a_{ij} can always be computed exactly if c is a positive constant, and the entries F_j can be computed exactly if the functions f and g are piecewise polynomials for example. If c is not constant, and f and g are general functions, then for computations of entries (which are sums of integrals over Ω and its boundary $\partial\Omega$) in system (8), we should use, in practice, special quadrature rules, and, thus, each such a case requires a separate analysis in the context of discrete maximum/minimum principles (cf. [10]).

Remark 6. Various geometric conditions on FE meshes guaranteeing the validity of the requirement $a_{ij} \le 0$ for $i \ne j$ are presented e.g. in [3, 8, 12].

4. DMPs for some finite difference (FD) schemes

On the base of several representative FD schemes, we shall demonstrate how the discrete maximum/minimum principles from Definition 4 can be proved also for finite difference approximations.

First, consider problem (1) posed in one-dimensional domain $\Omega = (0, 1)$. For the governing equation at the interior nodes we shall employ the following standard FD discretization:

$$\frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} + c_i y_i = f_i, \tag{19}$$

where $i = 1, ..., \hat{n} - 1$, $h = 1/\hat{n}$, c_i and f_i denote the values of functions c and f, respectively, at the node ih. The Neumann boundary condition is discretized as follows:

$$\frac{y_0 - y_1}{h} = g_0, \ \frac{y_{\hat{n}} - y_{\hat{n}-1}}{h} = g_{\hat{n}}. \tag{20}$$

The resulting FD system of linear equations is of size $(\hat{n} + 1) \times (\hat{n} + 1)$. However, its matrix is not SDD as, due to equations (20), the corresponding sums of entries of the matrix in the first and the last rows are zeros, so we cannot immediately use Theorem 3.

However, we notice that, e.g. under the sign-condition $g \le -g_\star < 0$ (used to prove the continuous maximum principle), it follows from (20) that $y_0 < y_1$ and $y_{\hat{n}} < y_{\hat{n}-1}$, and it is thus sufficient to get a suitable upper estimate only for the entries $y_1, \dots, y_{\hat{n}-1}$. Further, we form the reduced system of equations of the size $(\hat{n}-1)\times(\hat{n}-1)$ for finding (and estimating) $y_1, \dots, y_{\hat{n}-1}$ using discretization (19)–(20). This reduced system will consist of $\hat{n} - 3$ equations (19), for $i = 2, \dots, \hat{n} - 2$, and two following equations

$$\frac{y_1 - y_2}{h^2} + c_1 y_1 = f_1 + \frac{g_0}{h},$$
$$\frac{-y_{\hat{n}-2} + y_{\hat{n}-1}}{h^2} + c_{\hat{n}-1} y_{\hat{n}-1} = f_{\hat{n}-1} + \frac{g_{\hat{n}}}{h},$$

obtained by combining (20) and (19) for $i = 1$ and $i = \hat{n} - 1$. It is clear that the corresponding sums $\delta_i(\mathbf{A}) = c_i$, $i = 1, \dots, \hat{n} - 1$, and, therefore, the matrix of the reduced system is SDD and it is also M-matrix. Further, due to the sign-condition on g we observe that the entries of the right-hand side of the reduced system $F_i \le f_i$, $i = 1, \dots, \hat{n} - 1$. Therefore, estimates (11) and (12) are valid, i.e. the discrete maximum principle holds. The discrete minimum principle can be proved similarly under the condition $g \ge g_\star > 0$.

Consider now the two-dimensional case. Let, for simplicity, the solution domain be a square, i.e. $\Omega = (0, 1) \times (0, 1)$. Using the same step-size $h = 1/\hat{n}$ in both directions and the classical 5-point FD stencil, we arrive at the following interior equations inside of $\overline{\Omega}$

$$\frac{-y_{i-1,j} - y_{i+1,j} - y_{i,j-1} - y_{i,j+1} + 4y_{i,j}}{h^2} + c_{i,j} y_{i,j} = f_{i,j}, \tag{21}$$

where now $i, j = 1, ..., \hat{n}-1$ and $c_{i,j}$ and $f_{i,j}$ denote the values of functions c and f, respectively, at the node (ih, jh).

The first order accurate FD discretization of the Neumann boundary condition on $\partial\Omega$ (consisting of four intervals in this case) reads as follows:

$$\frac{y_{i,0} - y_{i,1}}{h} = g_{i,0}, \ \frac{y_{i,\hat{n}} - y_{i,\hat{n}-1}}{h} = g_{i,\hat{n}}, \ i = 1, 2, ..., \hat{n}-1, \tag{22}$$

$$\frac{y_{0,j} - y_{1,j}}{h} = g_{0,j}, \ \frac{y_{\hat{n},j} - y_{\hat{n}-1,j}}{h} = g_{\hat{n},j}, \ j = 1, 2, ..., \hat{n}-1, \tag{23}$$

where $g_{i,j}$ denotes the value of g at the node (ih, jh). We notice that we do not deal with the corner points of Ω in our case as the normal vectors are not well defined at these points.

We see again, that the matrix of the full system is not SDD, however, just the same trick as in the one-dimensional case can be used. And the following results can be easily proved.

Theorem 5. *The FD discretization* (21)–(23) *has the following properties:*

a) it approximates a sufficiently smooth solution u with the first order of accuracy,
b) the reduced FE matrix is SDD and is M-matrix,
c) the discrete maximum/minimum principles are valid provided the relevant conditions on g hold.

The approximation (22)–(23) (and (20)) of the Neumann boundary condition has only the first order of accuracy, which is not consistent with the second order of accuracy of the FD discretization for the governing differential equation. Therefore, we shall present and analyse another FD scheme, now with an increased accuracy of approximation for the Neumann boundary condition. We discuss in detail only the more complicated 2D case, because the analysis of 1D case is similar. So, let us approximate the Neumann boundary condition on the boundary of $\Omega = (0,1)\times(0,1)$ in the following manner:

- on the part of the boundary with $x = 0$ as

$$\begin{aligned} &\frac{y_{0,j} - y_{1,j}}{h} - \frac{h}{2}\left(\frac{y_{0,j+1} - 2y_{0,j} + y_{0,j-1}}{h^2}\right) + \frac{h}{2}c_{0,j}y_{0,j} = \\ &= g_{0,j} + \frac{h}{2}f_{0,j}, \quad j = 1, 2, \ldots, \hat{n}-1. \end{aligned} \tag{24}$$

- on the part of the boundary with $x = 1$ as

$$\begin{aligned} &\frac{y_{\hat{n},j} - y_{\hat{n}-1,j}}{h} - \frac{h}{2}\left(\frac{y_{\hat{n},j+1} - 2y_{\hat{n},j} + y_{\hat{n},j-1}}{h^2}\right) + \frac{h}{2}c_{\hat{n},j}y_{\hat{n},j} = \\ &= g_{\hat{n},j} + \frac{h}{2}f_{\hat{n},j}, \quad j = 1, 2, \ldots, \hat{n}-1. \end{aligned} \tag{25}$$

- on the part of the boundary with $y = 0$ as

$$\begin{aligned} &\frac{y_{i,0} - y_{i,1}}{h} - \frac{h}{2}\left(\frac{y_{i+1,0} - 2y_{i,0} + y_{i-1,0}}{h^2}\right) + \frac{h}{2}c_{i,0}y_{i,0} = \\ &= g_{i,0} + \frac{h}{2}f_{i,0}, \quad i = 1, 2, \ldots, \hat{n} - 1. \end{aligned} \tag{26}$$

- on the part of the boundary with $y = 1$ as

$$\begin{aligned} &\frac{y_{i,\hat{n}} - y_{i,\hat{n}-1}}{h} - \frac{h}{2}\left(\frac{y_{i+1,\hat{n}} - 2y_{i,\hat{n}} + y_{i-1,\hat{n}}}{h^2}\right) + \frac{h}{2}c_{i,\hat{n}}y_{i,\hat{n}} = \\ &= g_{i,\hat{n}} + \frac{h}{2}f_{i,\hat{n}}, \quad i = 1, 2, \ldots, \hat{n} - 1. \end{aligned} \tag{27}$$

Theorem 6. *The FD discretization* (21), (24)–(27) *has the following properties:*

a) it approximates a sufficiently smooth solution u with the second order of accuracy,
b) the resulting FD matrix $\mathbf{A}$ is SDD and M-matrix,
c) the discrete maximum/minimum principles are valid provided the relevant conditions on g hold.

Proof. We shall prove the statement a) only for the case of the part of the boundary with $x = 1$, because the proofs for the other cases are similar. Clearly, it is sufficient to show the second order of accuracy at the boundary nodes only. Let us define

$$\begin{aligned} \Psi_j =& \frac{u(1, jh) - u(1-h, jh)}{h} - \\ &- \frac{h}{2}\left(\frac{u(1,(j+1)h) - 2u(1, jh) + u(1,(j-1)h)}{h^2}\right) + \\ &+ \frac{h}{2}c(1, jh)u(1, jh) - g(1, jh) - \frac{h}{2}f(1, jh). \end{aligned} \tag{28}$$

Using the Taylor expansion, we get

$$\frac{u(1, jh) - u(1-h, jh)}{h} = \left(\partial_1 u - \frac{h}{2}\partial_{11}^2 u\right)\Big|_{(1,jh)} + \mathcal{O}(h^2), \tag{29}$$

$$\frac{u(1,(j+1)h) - 2u(1, jh) + u(1,(j-1)h)}{h^2} = \left(\partial_{22}^2 u\right)\big|_{(1,jh)} + \mathcal{O}(h^2), \tag{30}$$

where symbols like $\partial_i u$ and $\partial_{ij} u$ denote the partial derivatives of u as usual. Hence, putting (29) and (30) into (28), we obtain

$$\Psi_j = (\partial_1 u - g)\big|_{(1,jh)} - \frac{h}{2}\left(\partial_{11}^2 u + \partial_{22}^2 u - cu + f\right)\big|_{(1,jh)} + \mathcal{O}(h^2). \tag{31}$$

Due to the boundary condition in (1), and the relation $\frac{\partial u}{\partial n}(1, y) = \partial_1 u(1, y)$, the first term in the right-hand side of (31) vanishes. The second term is also equal to zero. This shows the validity of the statement a).

To prove the statement b), it is enough to show the diagonal dominance at the boundary nodes only. For convenience, we introduce the index k to have the single-index numbering of all the nodes of our domain (in order to keep the consistency with the "single-index" definition of $\delta_k(\mathbf{A})$) in which the indices $1, 2, \ldots, n^*$ are preserved for n^* interior nodes and the indices $n^* + 1, \ldots, n^* + n^0$ are used for n^0 boundary nodes. Then we have that

$$\delta_k(\mathbf{A}) = \frac{h}{2} c_k > 0 \quad \text{for} \quad k = n^* + 1, \ldots, n^* + n^0. \tag{32}$$

Therefore, under our assumptions $\mathbf{A}$ is SDD matrix and M-matrix.

To prove the statement c), one observes that for the right-hand side of the resulting FD system we have

$$F_k = f_k \quad \text{for} \quad k = 1, \ldots, n^*, \text{ and } F_k = g_k + \frac{h}{2} f_k \quad \text{for} \quad k = n^* + 1, \ldots, n^* + n^0. \tag{33}$$

Due to the property b), Theorem 3 can now be used. To get estimates (11) and (12), we use the corresponding sign-conditions on g. $\square$

5. Final remarks

It would be interesting to obtain suitable practical conditions guaranteeing the validity of our variant of discrete maximum/minimum principles also for various *hp*-versions of FEM (see [16]), and analyse the case of elliptic problems with full diffusive tensors (cf. [15]).

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