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DIMENSION REDUCTION FOR INCOMPRESSIBLE PIPE AND OPEN CHANNEL FLOW INCLUDING FRICTION

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Abstract: We present the full derivation of a one-dimensional free surface pipe or open channel flow model including friction with non constant geometry. The free surface model is obtained from the three-dimensional incompressible Navier-Stokes equations under shallow water assumptions with prescribed "well-suited" boundary conditions.

Keywords: free surface flow, incompressible Navier-Stokes, shallow water approximation, hydrostatic approximation, closed water pipe, open channel, friction

MSC: 65M08, 65M75, 76B07, 76M12, 76M28, 76N15

1. Introduction

Simulation of free surface pipe or open channel flow plays an important role in many engineering applications such as storm sewers, waste or supply pipes in hydroelectric installations, etc.

The free surface flows are described by a newtonian, viscous and incompressible fluid through the three-dimensional incompressible Navier-Stokes equations. The use of the full three-dimensional equations leads to time-consuming simulations. Therefore, for specific applications such as shallow water, one can proceed to a model reduction preserving some of the main physical features of the flow leading to the so-called shallow water equations. This is one of the most challenging issues that we address with the obvious consequence to decrease the computational time. During these last years, many efforts were devoted to the modelling and the simulation of free surface water flows (see for instance [14, 13, 6, 5, 10, 9, 8, 11, 1, 7, 2, 3] and the reference therein).

The classical shallow water equations are usually derived from the three-dimensional Navier-Stokes equations (or the two-dimensional Navier-Stokes equations) by vertical averaging. It leads to a two-dimensional or a one-dimensional shallow water model. For instance, Gerbeau and Perthame [10] study the full derivation of the one-dimensional shallow water equations from the two-dimensional Navier-Stokes equations while [11] considers the two-dimensional equations obtained from the three-dimensional one. In both cases, the so-called "motion by slices" is obtained. This property ensures that the horizontal velocity does not depend upon the vertical coordinate. As a consequence, one can perform the model reduction by vertical averaging. Following the applications under consideration, one can take into account as a source term the Coriolis effects, the topography, the friction, the capillary effects, the geometry, etc.

Unlike the previous works, we propose to study the full derivation of a **one-dimensional** free surface flows for pipe and open channel from the **three-dimensional** Navier-Stokes equations. In particular, we propose to revisit the work by Bourdarias et al. [3] done in the context of the three-dimensional Euler equations. The use of the Navier-Stokes equations with suitable boundary conditions allows first to establish the crucial "motion by slices" property, and second to include the friction (linear or non-linear) into the derivation. Let us emphasize that it was not possible to deal with in the framework of Bourdarias et al. [3]. More precisely, this property was assumed from the beginning and the friction was added to the obtained averaged equations.

The paper is organized as follows. In Section 2, we recall the full incompressible Navier-Stokes equations defining the boundary conditions including a general friction law, and we fix the notations. The "motion by slices" property under large Reynolds number flows is obtained through the hydrostatic equations (approximation) in Section 3. Next, these equations are averaged through the pipe or open channel section assumed to be orthogonal to the main flow direction. Finally, we obtain the one-dimensional free surface model. Since the constructed model is similar to the one by Bourdarias et al. [3], the issues of the numerical approximation is not addressed here. Please, refer to [1] or [4].

2. The incompressible Navier-Stokes equation and its closure

In this section, we fix the notations of the geometrical quantities involved to describe the thin domain representing a pipe or an open channel. In particular, without loss of generality (see Remark 2.1), we consider the case of pipe with circular section.

2.1. Geometrical settings

Let us consider an incompressible fluid confined in a three-dimensional rigid domain $\mathcal{P}$ representing a pipe or a channel, of length L:

$$\mathcal{P} := \big\{(x, y, z) \in \mathbb{R}^3;\ x \in [0, L],\ (y, z) \in \Omega_p(x)\big\}$$

where the section $\Omega_p(x)$, $x \in [0, L]$, is

$$\Omega_p(x) = \{(y, z) \in \mathbb{R}^2;\ y \in [\alpha(x, z), \beta(x, z)],\ z \in [0, 2R(x)]\}$$

as displayed on figure 1(a). Both flows and pipe are assumed to be oriented in the **i**-direction.

With these settings, we define the free surface section by

$$\Omega(t,x) = \Omega_p(x) \cap \{(y,z) \in \mathbb{R}^2; 0 \leqslant z \leqslant H(t,x,y)\}, \quad t > 0,\ x \in [0,L]$$

assumed to be orthogonal to the main flow direction. $H(t,x,y)$ is the local water elevation from the surface $z=0$ in the $\Omega_p(x)$-plane. $R(x)$ stands for the radius of the pipe section $S(x) = \pi R^2(x)$, $\alpha(x,z)$ (resp. $\beta(x,z)$) is the left (resp. the right) boundary point at elevation $0 \leqslant z \leqslant 2R(x)$ as displayed on figure 1(b).

On the wet boundary (part of the boundary in contact with water), we define the coordinate of a point $\mathbf{m} \in \partial\Omega(t,x) := \Gamma_b(t,x)$, $t > 0$, $x \in [0,L]$, by $(y, \varphi(x,y))$ where

$$\Gamma_b(t,x) = \{(y,z) \in \mathbb{R}^2;\ z = \varphi(x,y) \leqslant H(t,x,y)\}\ .$$

Then, we note $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$ the outward unit vector at the point $\mathbf{m} \in \partial\Omega(t,x)$, $x \in [0,L]$ as represented on figure 1(b). The point $\mathbf{m}$ also stands for the vector $\omega\mathbf{m}$ where $\omega(x,0,b(x))$ defines the main slope elevation of the pipe with $b'(x) = \sin\theta(x)$.

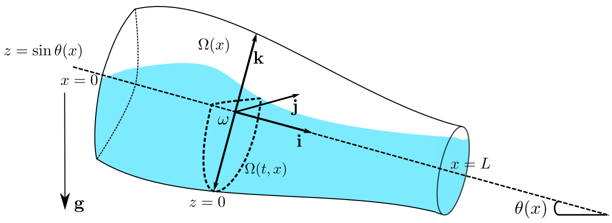
On the free surface, we define the coordinate of a point $\mathbf{m} \in \partial\Omega(t,x) := \Gamma_{fs}(t,x)$, $t > 0$, $x \in [0,L]$, by $(y, H(t,x,y))$ where

$$\Gamma_{fs}(t,x) = \{(y,z) \in \mathbb{R}^2;\ z = H(t,x,y)\}\ .$$

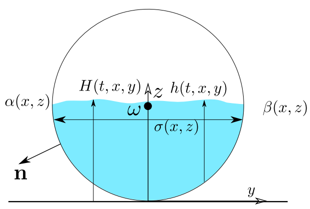
Finally, we note

$$h(t,x,y) = H(t,x,y) - \varphi(x,y)$$

the local elevation of the water.



(a) Configuration



(b) Ω-plane

Figure 1: Geometric characteristics of the pipe

Remark 2.1 One can easily adapt this work to any realistic pipe or open channel by defining appropriately the previous quantities. For instance, in the case of "horseshoe" section (see figure 2(a)), the section $\Omega_p(x)$, $x \in [0,L]$, is given by

$$\Omega_p(x) = \Omega_H(x) \cap \Omega_R(x)$$

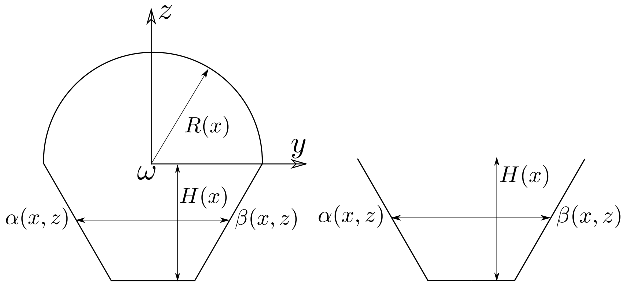
where

$$\Omega_H(x) = \left\{(y,z) \in \mathbb{R}^2; y \in [\alpha(x,z), \beta(x,z)],\ z \in [0, H(x)]\right\}$$

and

$$\Omega_R(x) = \{(y,z) \in \mathbb{R}^2;\ y \in [\alpha(x,z), \beta(x,z)],\ z \in [H(x), R(x)]\}\ .$$

H is the height of the trapezoidal basis and R is the radius of the upper part of the "horseshoe". A second example is represented on figure 2(b).



(a) "horseshoe" section (b) open channel

Figure 2: Example of a pipe and a open channel geometry

2.2. The water flow model

In the domain $\mathcal{P}$, we assume that the flow is incompressible and the pipe is always partially filled (otherwise we have to deal with pressurized flows that we omit here, please see [3] for details). Thus, we consider the incompressible Navier-Stokes equations with a prescribed general wall law conditions including friction on the wet boundary and a no stress one on the free surface. We complete the system with inflows and outflows conditions at the upstream and downstream ends.

The governing equations for the motion of an incompressible fluid in $[0,T] \times \mathcal{P}$, $T > 0$ are given by

$$\left\{ \begin{array}{rcl} \mathrm{div}(\rho_0 \mathbf{u}) & = & 0\ , \\ \partial_t(\rho_0 \mathbf{u}) + \mathrm{div}(\rho_0 \mathbf{u} \otimes \mathbf{u}) - \mathrm{div}\sigma - \rho_0 F & = & 0\ , \end{array} \right. \tag{1}$$

where $\mathbf{u} = \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix}$ is the velocity fields with u the **i**-component and $\mathbf{v} = \begin{pmatrix} v \\ w \end{pmatrix}$ the Ω-component, ρ_0 is the density of the fluid at atmospheric pressure and $F = -g \begin{pmatrix} -\sin\theta(x) \\ 0 \\ \cos\theta(x) \end{pmatrix}$ is the external gravity force of constant g. The total stress tensor can be written:

$$\sigma = \begin{pmatrix} -p + 2\mu\partial_x u & \mathcal{R}(\mathbf{u})^t \\ \mathcal{R}(\mathbf{u}) & -pI_2 + 2\mu D_{y,z}(\mathbf{v}) \end{pmatrix} \tag{2}$$

where I_2 is the identity matrix, μ is the dynamical viscosity and $\mathcal{R}(\mathbf{u})$ is defined by $\mathcal{R}(\mathbf{u}) = \mu\left(\nabla_{y,z}u + \partial_x \mathbf{v}\right)$. $\nabla_{y,z}u = \begin{pmatrix} \partial_y u \\ \partial_z u \end{pmatrix}$ is the gradient of u with respect to (y, z). Noting X^t the transpose of X, we define the strain tensor $D_{y,z}(\mathbf{v})$ with respect to the variable (y, z):

$$2D_{y,z}(\mathbf{u}) = \nabla_{y,z}\mathbf{v} + \nabla^t_{y,z}\mathbf{v}\ .$$

2.3. The boundary conditions

The Navier-Stokes system (1)–(2) is completed with suitable boundary conditions to introduce the border friction term on the wet boundary. On the free surface, we prescribe a no-stress condition.

On the wet boundary

For pipe flow calculations, the Darcy-Weisbach equation, valid for laminar as well as turbulent flows, is generally adopted. Roughly speaking, such formula relates losses h occurred during flows and it reads:

$$h = C_f \frac{L}{D}\frac{U^2}{2g}$$

where L, D, U are the pipe length, the pipe diameter and the velocity. The friction factor C_f, rather being a simple constant, turns out to be a factor that depends upon several parameters such as the Reynolds number R_e, the relative roughness δ, the Froude number F_r, the Mach number M_a, geometrical parameters, etc., and cannot be set as a constant. Following the type of the material, rough or smooth pipe, leaves C_f depend upon less quantities and lead to several expressions. An empirical transition function for the region between smooth pipes and the complete turbulence zone has been proposed by Colebrook:

$$\frac{1}{\sqrt{C_f}} = -0.86\ln\left(\frac{\delta}{3.7D} + \frac{2.51}{R_e\sqrt{C_f}}\right)$$

where δ is the roughness of the material.

Because of the extreme complexity of the rough surfaces, most of the advances in understanding have been developed around experiments leading to charts such as the Moody-Stanton diagram, expressing C_f as a function of the Reynolds number R_e, the relative roughness and some geometrical parameters depending on the material. This yields to several formula depending on the modelling, for instance Chézy and Manning which are well-known by the engineers community, see for instance [16, 15].

For laminar flow, the effects of the material roughness can be ignored due to a presence of a thin laminar film at the pipe wall. Then, it can be shown that the Darcy-Weisbach equation reduces to $C_f = \dfrac{64}{R_e}$ that we note $C_f = C_l$ in the sequel. And, the losses are directly proportional to the velocity. When increasing the

Reynolds number R_e, the thin laminar film becomes unstable and causes turbulence increasing the head loss. Thus, the dependence on the Reynolds number R_e can be neglected and the head loss is almost directly proportional to U^2. The value of the friction factor, that we note $C_f = C_t$ in the sequel, can be read on diagrams.

In particular, this motivates the use of the following general friction law:

$$k(\mathbf{u})\mathbf{u} = C_f(|\mathbf{u}|)\mathbf{u} = C_l\mathbf{u} + C_t|\mathbf{u}|\mathbf{u},\ C_l \geqslant 0, C_t > 0 \tag{3}$$

where C_f stands for the friction factor. We do not intend in this work to define precisely the friction law but instead, we want to directly include it in its general form to explicitly show its dependency on physical parameters in the present model reduction.

Thus, on the inner wall $\partial\Omega_p(x)$, $\forall x \in (0, L)$, we assume a wall-law condition including a general friction law:

$$(\sigma(\mathbf{u})\mathbf{n}_b) \cdot \boldsymbol{\tau}_{b_i} = \rho_0 k(\mathbf{u})\mathbf{u} \cdot \boldsymbol{\tau}_{b_i},\ \ x \in (0, L),\ (y, z) \in \Gamma_b(x),\ i = 1, 2$$

where $\boldsymbol{\tau}_{b_i}$ is the i^{th} vector of the tangential basis and $\mathbf{n}_b$ stands for the unit outward normal vector:

$$\mathbf{n}_b = \frac{1}{\sqrt{(\partial_x\varphi)^2 + \mathbf{n}\cdot\mathbf{n}}}\begin{pmatrix} -\partial_x\varphi \\ \mathbf{n} \end{pmatrix}$$

with $\mathbf{n} = \begin{pmatrix} -\partial_y\varphi \\ 1 \end{pmatrix}$ the outward normal vector in the Ω_p-plane. Writing the wall-law condition in its vectorial form (i.e. the tangential constraints),

$$\sigma(\mathbf{u})\mathbf{n}_b - (\sigma(\mathbf{u})\mathbf{n}_b \cdot \mathbf{n}_b)\,\mathbf{n}_b = \rho_0 k(\mathbf{u})\mathbf{u},\ \ t > 0,\ x \in (0, L),\ (y, z) \in \Gamma_b(t, x)\ ,$$

one can split up the $\mathbf{i}-$component and the $(\mathbf{j}, \mathbf{k})-$components. Thus, the wall-law boundary conditions are

$$\begin{aligned}
\mathcal{R}(\mathbf{u}) \cdot \mathbf{n}\ (\mathbf{n}\cdot\mathbf{n} - (\partial_x\varphi)^2) + 2\mu\partial_x\varphi\,(D_{y,z}(\mathbf{v})\mathbf{n}\cdot\mathbf{n} - \partial_x u\ (\mathbf{n}\cdot\mathbf{n}))& \\
= \left(\mathbf{n}\cdot\mathbf{n} + (\partial_x\varphi)^2\right)^{3/2}\rho_0 k(u)u\ ,& \quad (4)\\
2\mu(\partial_x\varphi)^2\,(D_{y,z}(\mathbf{v})\mathbf{n} - \mathbf{n}) + \partial_x\varphi\mathcal{R}(\mathbf{u})\ (\mathbf{n}\cdot\mathbf{n} - (\partial_x\varphi)^2)& \\
= \left(\mathbf{n}\cdot\mathbf{n} + (\partial_x\varphi)^2\right)^{3/2}\rho_0 k(\mathbf{v})\mathbf{v}\ .& \quad (5)
\end{aligned}$$

supplemented with a no-penetration condition:

$$\mathbf{u}\cdot\mathbf{n}_b = 0,\ \ t > 0,\ x \in (0, L),\ (y, z) \in \Gamma_b(t, x)$$

i.e.

$$u\partial_x\varphi = \mathbf{v}\cdot\mathbf{n},\ \ t > 0,\ x \in (0, L),\ (y, z) \in \Gamma_b(t, x)\ . \tag{6}$$

On the free surface boundary

For the sake of simplicity, on the free surface we assume a no-stress condition:

$$\sigma(\mathbf{u})\boldsymbol{N}^{fs} = 0,\ \ t > 0,\ \ x \in (0, L),\ \ (y, z) \in \Gamma_{fs}(t, x)$$

where

$$\boldsymbol{N}^{fs} = \frac{1}{\sqrt{(\partial_x H)^2 + \mathbf{n}_{fs} \cdot \mathbf{n}_{fs}}} \begin{pmatrix} -\partial_x H \\ \mathbf{n}_{fs} \end{pmatrix} \text{ where } \mathbf{n}_{fs} = \begin{pmatrix} -\partial_y H \\ 1 \end{pmatrix}$$

is the outward normal vector to the free surface.

Finally, as done before, splitting up the horizontal and the Ω_p-component, the free surface boundary conditions read

$$(p - 2\mu\partial_x u)\partial_x H + R(u) \cdot \mathbf{n}_{fs} = 0\ , \tag{7}$$

$$R(u)\partial_x H + (p - 2\mu D_{y,z}(\mathbf{v}))\mathbf{n}_{fs} = 0\ . \tag{8}$$

Introducing the indicator function Φ of the fluid region

$$\Phi(t, x, y, z) = \begin{cases} 1 & \text{if } \varphi(x, y) \leqslant z \leqslant H(t, x, y)\ , \\ 0 & \text{otherwise} \end{cases}$$

and because of the incompressibility condition, the divergence equation can be expressed as follows:

$$\partial_t \Phi + \partial_x(\Phi u) + \mathrm{div}_{y,z}(\Phi \mathbf{v}) = 0\ . \tag{9}$$

3. The averaged model

The technique presented in this section is the one introduced by Gerbeau and Perthame [10] in the context of the reduction of the two-dimensional incompressible Navier-Stokes model to the one-dimensional shallow water equations. Here, instead, we proceed to the reduction of the three-dimensional incompressible Navier-Stokes equations to a one-dimensional shallow water equations.

3.1. Dimensionless Navier-Stokes equations

Thus, in the sequel we consider the non-dimensional form of the Navier-Stokes system using the shallow water assumption by introducing a "small" parameter so that

$$\varepsilon = \frac{D}{L} = \frac{W}{U} = \frac{V}{U} \ll 1$$

where U, $\boldsymbol{V} = (V, W)$ are the characteristic speeds in the $\mathbf{i}-$direction and the $(\mathbf{j}, \mathbf{k})-$direction.

We introduce a characteristic time T and a characteristic pressure P such that $T = \dfrac{L}{U}$ and $P = \rho_0 U^2$. The dimensionless quantities of time $\tilde{t}$, coordinate $(\tilde{x}, \tilde{y}, \tilde{z})$ and velocity field $(\tilde{u}, \tilde{v}, \tilde{w})$, noted temporarily by a $\tilde{\cdot}$, are defined by

$$\tilde{t} = \frac{t}{T}, \quad (\tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{x}{L}, \frac{y}{D}, \frac{z}{D}\right), \quad (\tilde{u}, \tilde{v}, \tilde{w}) = \left(\frac{u}{U}, \frac{v}{W}, \frac{w}{W}\right)$$

with the modified friction factor C_f/U that we write in the sequel C_f.

Let us define the following non-dimensional numbers:

F_r	Froude number following the Ω-plane	: $F_r = U/\sqrt{gD}$,
F_L	Froude number following the **i**-direction	: $F_L = U/\sqrt{gL}$,
R_e	Reynolds number with respect to μ	: $R_e = \rho_0 UL/\mu$.

Using these new variables in Equations (1), dropping the $\tilde{\cdot}$, ordering the terms with respect to ε, the dimensionless incompressible Navier-Stokes system becomes:

$$\begin{aligned}
\mathrm{div}(\mathbf{u}) &= 0 &(10)\\
\partial_t(u) + \partial_x(u^2) + \mathrm{div}_{y,z}(u\mathbf{v}) + \partial_x p &= -\frac{\sin\theta(x)}{F_L^2} + \mathrm{div}_{y,z}\left(\frac{R_e^{-1}}{\varepsilon^2}\nabla_{y,z} u\right) &(11)\\
&\quad + R_{\varepsilon,1}(\mathbf{u})\\
\nabla_{y,z} p &= \begin{pmatrix} 0 \\ -\dfrac{\cos\theta(x)}{F_r^2} \end{pmatrix} + R_{\varepsilon,2}(\mathbf{u}) &(12)
\end{aligned}$$

where

$$R_{\varepsilon,1}(\mathbf{u}) = R_e^{-1}\left(\partial_x\left(2\partial_x u\right) + \mathrm{div}_{y,z}\left(\partial_x \mathbf{v}\right)\right) = O(R_e^{-1})$$

and

$$\begin{aligned}
R_{\varepsilon,2}(\mathbf{u}) &= R_e^{-1}\left(\partial_x\left(\nabla_{y,z} u + \varepsilon^2\partial_x\mathbf{v}\right) + \mathrm{div}_{y,z}\left(2D_{y,z}(\mathbf{v})\right)\right)\\
&\quad -\varepsilon^2\left(\partial_t(\mathbf{v}) + \partial_x(u\mathbf{v}) + \mathrm{div}_{y,z}(\mathbf{v}\otimes\mathbf{v})\right)\ ,\\
&= R_e^{-1}\left(\partial_x\left(\nabla_{y,z} u\right) + \mathrm{div}_{y,z}\left(2D_{y,z}(\mathbf{v})\right)\right) + \boldsymbol{O}(\varepsilon^2)\ ,\\
&= \boldsymbol{O}(R_e^{-1}) + \boldsymbol{O}(\varepsilon^2)\ .
\end{aligned}$$

The first component of the wall-law boundary condition (4) becomes:

$$\begin{aligned}
\frac{R_e^{-1}}{\varepsilon}\nabla_{y,z} u\cdot\mathbf{n} &= \frac{\left(\mathbf{n}\cdot\mathbf{n} + \varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\frac{k(u)}{U}u}{\left(\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2\right)}\\
&\quad \varepsilon R_e^{-1}\left(\frac{2\partial_x\varphi\left(D_{y,z}(\mathbf{v})\mathbf{n}\cdot\mathbf{n} - \partial_x u\ (\mathbf{n}\cdot\mathbf{n})\right)}{\left(\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2\right)} + \partial_x\mathbf{v}\cdot\mathbf{n}\right)\ , &(13)\\
&= -K(u) + O(\varepsilon) + O(\varepsilon R_e^{-1})
\end{aligned}$$

where we make use of the notations

$$K(u) = \sqrt{\mathbf{n}\cdot\mathbf{n}}\frac{k(u)}{U}u \quad \text{and} \quad \nabla_{y,z} u\cdot\mathbf{n} := \partial_{\mathbf{n}} u$$

which are respectively the friction term and the normal derivative of u in the Ω_p-plane.

The second component of the wall-law boundary condition (5) becomes:

$$\begin{aligned} R_e^{-1}\nabla_{y,z}u &= \frac{\varepsilon^2\left(\mathbf{n}\cdot\mathbf{n}+\varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\rho_0\frac{k(\mathbf{v})}{U}\mathbf{v}}{\partial_x\varphi(\mathbf{n}\cdot\mathbf{n}-\varepsilon^2(\partial_x\varphi)^2)} \\ &\quad -\frac{2\varepsilon^3R_e^{-1}\partial_x\varphi^2\left(D_{y,z}(\mathbf{v})\mathbf{n}-\mathbf{n}\right)}{\partial_x\varphi(\mathbf{n}\cdot\mathbf{n}-\varepsilon^2(\partial_x\varphi)^2)}-\varepsilon^2\partial_x\mathbf{v}\cdot\mathbf{n}\ , \\ &= \boldsymbol{O}(\varepsilon^2)+\boldsymbol{O}(\varepsilon^3R_e^{-1}) \end{aligned} \tag{14}$$

On the free surface, the boundary conditions (7)-(8) are now

$$\begin{aligned} R_e^{-1}\nabla_{y,z}u\cdot\mathbf{n}_{fs} &= -\varepsilon^2\left((p-2R_e^{-1}\partial_xu)\partial_xH+R_e^{-1}\partial_x\mathbf{v}\cdot\mathbf{n}_{fs}\right) &(15)\\ &= O(\varepsilon^2)\ , \\ (p-2R_e^{-1}D_{y,z}(\mathbf{v}))\mathbf{n}_{fs} &= -\left(R_e^{-1}\nabla_{y,z}u+\varepsilon^2R_e^{-1}\partial_x\mathbf{v}\right)\partial_xH\ . &(16) \end{aligned}$$

Thanks to the relations (15) and (16), the pressure on the free surface satisfies the following equality

$$p\ (\mathbf{n}_{fs}\cdot\mathbf{n}_{fs})-2R_e^{-1}D_{y,z}(\mathbf{v})\mathbf{n}_{fs}\cdot\mathbf{n}_{fs}=\varepsilon^2\left(\partial_xH\right)^2(p-2R_e^{-1}\partial_xu)=O(\varepsilon^2)\ . \tag{17}$$

3.2. First order approximation

As emphasized before in Section 2.3, when increasing the Reynolds number R_e, we observe instabilities at the pipe wall leading to turbulent flows. Assuming the characteristic length of the thin unstable film is larger than the relative roughness of the pipe, one can always assume some smallness of the friction law (see for instance [16, 15]). In particular, it motivates, for large Reynolds number R_e, the following asymptotic assumptions:

$$R_e^{-1}=\varepsilon\mu_0,\quad K=\varepsilon K_0 \tag{18}$$

where μ_0 is some viscosity constant and K_0 is the asymptotic friction law

$$K_0(u)=\sqrt{\mathbf{n}\cdot\mathbf{n}}\ k(u)u\ . \tag{19}$$

Under these conditions, the Archimedes principle is applicable and induces small vertical accelerations. As a consequence, one can drop all terms of order $O(\varepsilon^2)$ in Equations (10)–(12). Then, taking the formal limit as ε goes to 0, we deduce the hydrostatic equations

$$\begin{aligned} \partial_x(u_\varepsilon)+\mathrm{div}_{y,z}(\boldsymbol{v}_\varepsilon) &= 0 &(20)\\ \partial_t(u_\varepsilon)+\partial_x(u_\varepsilon^2)+\mathrm{div}_{y,z}(u_\varepsilon\boldsymbol{v}_\varepsilon)+\partial_xp_\varepsilon &= -\frac{\sin\theta(x)}{F_L^2}+\mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right) &(21)\\ \nabla_{y,z}p_\varepsilon &= \begin{pmatrix}0\\ -\dfrac{\cos\theta(x)}{F_r^2}\end{pmatrix} &(22) \end{aligned}$$

Let us emphasize that even if this system results from a formal limit, we note its solution $(p_\varepsilon, u_\varepsilon, \boldsymbol{v}_\varepsilon)$ due to the explicit dependency on ε in the term $\mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right)$ in Equation (21). At zero order, this term will be precisely the friction at the wet boundary through the condition (13). In particular, the boundary conditions write

- on the wet boundary; conditions (13)-(14) are

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon \cdot \mathbf{n} = K_0(u_\varepsilon) + O(\varepsilon), \qquad t>0,\ x\in(0,L),\ (y,z)\in\Gamma_b(t,x)\ . \tag{23}$$

- on the free surface boundary; conditions (15)-(16) and (17) are

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon \cdot \boldsymbol{n}_\varepsilon^{fs} = O(\varepsilon), \qquad t>0,\ x\in(0,L),\ (y,z)\in\Gamma_{fs}(t,x)\ . \tag{24}$$

Next, identifying terms at order $\frac{1}{\varepsilon}$ in Equations (20)–(22), thanks to Equations (23) and (24), we obtain the so-called "motion by slices"

$$u_\varepsilon(t,x,y,z) \;=\; u_0(t,x) + O(\varepsilon) \tag{25}$$

for some function $u_0 = u_0(t,x)$, by solving formally the Neumann problem for $t>0$, $x\in(0,L)$

$$\left\{\begin{array}{lcl} \mathrm{div}_{y,z}\left(\mu_0\nabla_{y,z}u_\varepsilon\right) & = & O(\varepsilon)\ ,\quad (y,z)\in\Omega(t,x) \\ \mu_0\partial_{\mathbf{n}}u_\varepsilon & = & O(\varepsilon)\ ,\quad (y,z)\in\partial\Omega(t,x) \end{array}\right.$$

One one hand, the following approximation at first order holds

$$u_\varepsilon(t,x,y,z) \;\approx\; \overline{u_\varepsilon}(t,x)$$

where $\overline{u_\varepsilon}(t,x) = \dfrac{1}{|\Omega_\varepsilon(t,x)|}\displaystyle\int_{\Omega_\varepsilon(t,x)} u_\varepsilon(t,x,y,z)\,dy\,dz$ is the mean speed of the fluid over the wet section. Consequently, one can approximate at first order the non-linear term as follows

$$\overline{u_\varepsilon^2} \approx \overline{u_\varepsilon}^2\ . \tag{26}$$

On the other hand, using the second component of Equations (22), we may write

$$\partial_z p_\varepsilon(t,x,y,z) = -\frac{\cos\theta(x)}{F_r^2} + O(\varepsilon)\ .$$

Then, fixing y and integrating this equation for $\xi\in[z,H(t,x,y)]$, keeping in mind the identity (17), we obtain

$$p_\varepsilon(t,x,y,z) = \frac{\cos\theta}{F_r^2}(H_\varepsilon(t,x,y)-z) + O(\varepsilon)\ .$$

Moreover, using the first component of Equations (22) leads to

$$H_\varepsilon(t,x,y) = H_\varepsilon(t,x,0) + O(\varepsilon) \; . \tag{27}$$

As a consequence, we recover the classical hydrostatic pressure

$$p_\varepsilon(t,x,y,z) \;\approx\; \frac{\cos\theta}{F_r^2}(H_\varepsilon(t,x,0) - z) \; , \tag{28}$$

Finally, in view of the the definition of the water elevation H_ε (27), the wet section is approximated at first order as follows, $t > 0, x \in [0, L]$:

$$\Omega_\varepsilon(t,x) = \{(y,z) \in \mathbb{R}^2; \alpha(x,z) \leqslant y \leqslant \beta(x,z) \text{ and } 0 \leqslant z \leqslant H_\varepsilon(t,x,0)\} \tag{29}$$

and the outward unit normal vector to the free surface $\mathbf{n}_{fs}$ is now $\boldsymbol{n}_\varepsilon^{fs} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as displayed on figure 3.

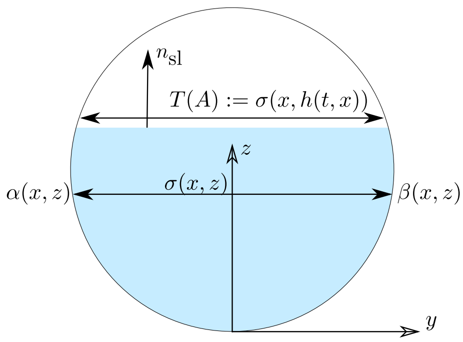


Figure 3: First order approximation of the wet area

In the sequel, due to its dependency at first order, we write $H_\varepsilon(t,x,y)$ by $H_\varepsilon(t,x)$.

3.3. The free surface model

By virtue of the relations (25)–(29), integrating Equations (20)–(22) over the cross-section $\Omega_p(t,x)$, the free surface model immediately follows.

First, let us recall that $\mathbf{m} = (y, \varphi(x,y)) \in \partial\Omega_p(x)$ stands for the vector $\omega\mathbf{m}$ and $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$ for the outward unit normal vector to the boundary Γ_b at the point $\mathbf{m}$ in the Ω_p-plane as displayed on figure 1(b).

Second, let us introduce $A(t,x)$ and $Q(t,x)$ the conservative variables of wet area and discharge defined by the following relations:

$$A(t,x) = \int_{\Omega_\varepsilon(t,x)} dydz \tag{30}$$

and

$$Q(t,x) = A(t,x)\overline{u_\varepsilon}(t,x) \tag{31}$$

where

$$\overline{u_\varepsilon}(t,x) = \frac{1}{A(t,x)} \int_{\Omega_\varepsilon(t,x)} u(t,x,y,z)\, dydz$$

is the mean speed of the fluid over the section $\Omega_\varepsilon(t,x)$.

Equation of the conservation of the momentum and the kinematic boundary condition

Let $\mathbf{v}$ be the vector field $\begin{pmatrix} v \\ w \end{pmatrix}$. Integrating the equation of conservation of the mass (9) on the set:

$$\overline{\Omega}(x) = \{(y,z);\, \alpha(x,z) \leqslant y \leqslant \beta(x,z),\, 0 \leqslant z \leqslant \infty\},$$

we get the following equation:

$$\int_{\overline{\Omega}(x)} \partial_t(\phi) + \partial_x(\phi u_\varepsilon) + \mathrm{div}_{y,z}(\phi \boldsymbol{v}_\varepsilon)\, dydz = \partial_t A + \partial_x Q - \int_{\partial\Omega_\varepsilon(t,x)} (u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \mathbf{n}\; ds\ . \tag{32}$$

Now, integrating Equation (9) on $\Omega_\varepsilon(t,x)$, we get:

$$\int_0^{H_\varepsilon(t,x)} \partial_t \int_{\alpha(x,z)}^{\beta(x,z)} dydz + \partial_x Q + \int_{\partial\Omega_\varepsilon(t,x)} (\boldsymbol{v}_\varepsilon - u_\varepsilon \partial_x \mathbf{m}) \cdot \mathbf{n}\, ds = 0$$

where

$$\int_0^{H_\varepsilon(t,x)} \partial_t \int_{\alpha(x,z)}^{\beta(x,z)} dydz = \partial_t A - \sigma(x, H_\varepsilon(t,x))\partial_t h$$

with $\sigma(x, H_\varepsilon(t,x))$ is the width at the free surface elevation as displayed on figure 3.

Then, one has:

$$\begin{aligned}
&\partial_t(A) + \partial_x(Q) - \int_{\Gamma_\varepsilon^{fs}(t,x)} (\partial_t \mathbf{m} + u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds \\
&- \int_{\Gamma_b(t,x)} (u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \mathbf{n}\, ds = 0.
\end{aligned} \tag{33}$$

Keeping in mind the no penetration condition (6) and comparing Equations (32) and (33), we finally derive the kinematic boundary condition at the free surface:

$$\int_{\Gamma_\varepsilon^{fs}(t,x)} (\partial_t \mathbf{m} + u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds = 0 \tag{34}$$

i.e.

$$\partial_t H_\varepsilon + u_\varepsilon(z = H_\varepsilon)\partial_x H_\varepsilon - w_\varepsilon(z = H_\varepsilon) = 0\ .$$

Finally, gathering Equations (33) and (34), we get the equation of the conservation of the mass:

$$\partial_t(A) + \partial_x(Q) = 0. \tag{35}$$

Equation of the conservation of the momentum

In order to get the equation of the conservation of the momentum of the free surface model, we integrate each term of Equation (21) over sections $\Omega_\varepsilon(t,x)$ as follows:

$$\int_{\Omega_\varepsilon(t,x)} \underbrace{\partial_t(u_\varepsilon)}_{a_1} + \underbrace{\partial_x(u_\varepsilon^2)}_{a_2} + \underbrace{\mathrm{div}_{y,z}\,(u_\varepsilon \boldsymbol{v}_\varepsilon)}_{a_3} + \underbrace{\partial_x p_\varepsilon}_{a_4}\, dydz = \int_{\Omega_\varepsilon(t,x)} -\underbrace{\frac{\sin\theta}{F_L^2}}_{a_5}\, dydz +$$

$$\int_{\Omega_\varepsilon(t,x)} \underbrace{\mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon\right)}_{a_6}\, dydz\ .$$

By virtue of relations (25), (26) and (28), we successively get:

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_1\, dydz$

The pipe being non-deformable, only the integral at the free surface is non zero since

$$\int_{\Gamma_b(t,x)} u_\varepsilon\, \partial_t \mathbf{m}\cdot\mathbf{n}\, ds = 0.$$

Thus, we get:

$$\int_{\Omega_\varepsilon(t,x)} \partial_t(u_\varepsilon)\, dydz \;=\; \partial_t \int_{\Omega_\varepsilon(t,x)} u_\varepsilon\, dydz - \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon\, \partial_t \mathbf{m}\cdot \boldsymbol{n}_\varepsilon^{fs}\, ds.$$

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_2\, dydz$

$$\begin{aligned}\int_{\Omega_\varepsilon(t,x)} \partial_x(u_\varepsilon^2)\, dydz \;&=\; \partial_x \int_{\Omega_\varepsilon(t,x)} u_\varepsilon^2\, dydz - \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon^2 \partial_x \mathbf{m}\cdot \boldsymbol{n}_\varepsilon^{fs}\, ds \\ &\quad - \int_{\Gamma_b(t,x)} u_\varepsilon^2 \partial_x \mathbf{m}\cdot\mathbf{n}\, ds.\end{aligned}$$

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_3\, dydz$

$$\int_{\Omega_\varepsilon(t,x)} \mathrm{div}_{y,z}\,(u_\varepsilon \boldsymbol{v}_\varepsilon)\ dydz \;=\; \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon \mathbf{v}\cdot \boldsymbol{n}_\varepsilon^{fs}\, ds + \int_{\Gamma_b(t,x)} u_\varepsilon \boldsymbol{v}_\varepsilon \cdot \mathbf{n}\, ds.$$

Summing the result of the previous step $a_1 + a_2 + a_3$, we get:

$$\int_{\Omega_\varepsilon(t,x)} a_1 + a_2 + a_3\, dydz = \partial_t(Q) + \partial_x\left(\frac{Q^2}{A}\right) \tag{36}$$

where A and Q are given by (30) and (31).

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_4\, dydz$

For the pressure term p_ε given by the relation (28), (t,x) fixed, we have:

$$\begin{aligned}
\int_{\Omega_\varepsilon(t,x)} \partial_x p_\varepsilon \, dydz &= \int_0^{H_\varepsilon(t,x)} \int_{\alpha(x,z)}^{\beta(x,z)} \partial_x p_\varepsilon \, dydz \\
&= \int_0^{H_\varepsilon(t,x)} \sigma(x,z)\partial_x p_\varepsilon \, dydz \\
&= \int_0^{H_\varepsilon(t,x)} \partial_x \left(p_\varepsilon \sigma(x,z)\right) dz - \int_0^{H_\varepsilon(t,x)} p_\varepsilon \partial_x \sigma(x,z)\, dz \\
&= \partial_x \int_{\Omega_\varepsilon(t,x)} p_\varepsilon \sigma(x,z)\, dydz \\
&\quad - \int_0^{H_\varepsilon(t,x)} p_\varepsilon \partial_x \sigma(x,z)\, dz - \partial_x H_\varepsilon(t,x) p_{\varepsilon|z=H_\varepsilon(t,x)}
\end{aligned}$$

Finally, we have:

$$\int_{\Omega_\varepsilon(t,x)} \partial_x p_\varepsilon \, dydz = \partial_x \left(g I_1(x,A) \frac{\cos\theta(x)}{F_r^2} \right) - g I_2(x,A) \frac{\cos\theta(x)}{F_r^2} \tag{37}$$

where I_1 is the hydrostatic pressure:

$$I_1(x,A) = \int_0^{H_\varepsilon(A)} (H_\varepsilon(A) - z)\sigma(x,z)\, dz.$$

The term I_2 is the pressure source term:

$$I_2(x,A) = \int_0^{H_\varepsilon(A)} (H_\varepsilon(A) - z)\partial_x\sigma(x,z)\, dz.$$

which takes into account of the section variation through the term $\partial_x \sigma(x,\cdot)$.

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_5\, dydz$

We have:

$$\int_{\Omega_\varepsilon(t,x)} g\sin\theta \, dydz = gA\sin\theta. \tag{38}$$

Computation of the term $\displaystyle\int_{\Omega_\varepsilon(t,x)} a_6\, dydz$

We have:

$$\int_{\Omega_\varepsilon(t,x)} \mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon\right) dydz = \int_{\Gamma_\varepsilon^{fs}(t,x)} \frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds + \int_{\Gamma_b(t,x)} \frac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon \cdot \mathbf{n}\, ds \tag{39}$$

where $\displaystyle\int_{\Gamma_\varepsilon^{fs}(t,x)} \frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon \cdot \boldsymbol{n}_\varepsilon^{fs}\,ds = 0$ due to the boundary condition (24). Using the boundary conditions (23) and the approximation (25), the second integral writes

$$\int_{\Gamma_b(t,x)} \frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon \cdot \mathbf{n}\,ds = \int_{\Gamma_b(t,x)} K_0(u_\varepsilon)\,ds = AK(\overline{u_\varepsilon})$$

where

$$K(x,\overline{u_\varepsilon}) = K_0(\overline{u_\varepsilon})\frac{\int_{\Gamma_b(t,x)} ds}{A}ds$$

with $\displaystyle\int_{\Gamma_b(t,x)} ds$ is the wet perimeter P_m (i.e. the portion of the perimeter where the wall is in contact with the fluid) and thus $\dfrac{A}{\int_{\Gamma_b(t,x)} ds}$ is nothing but the so-called hydraulic radius. This quantity was introduced by engineers as a length scale for non-circular ducts in order to use the analysis derived for the circular pipes (see for instance [16, 17]). Let us outline that this factor is naturally obtained in the derivation of the averaged model and holds for any realistic pipe or open channel (see Remark 2.1).

Then, gathering results (35) and (36)–(39), we get the equation of the conservation of the momentum. Finally, multiplying by $\rho_0 U^2/L$, the shallow water equations for free surface flows are:

$$\left\{\begin{array}{lcl} \partial_t(A) + \partial_x(Q) & = & 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + gI_1\cos\theta\right) & = & -gA\sin\theta + gI_2\cos\theta - gAK(x,Q/A) \end{array}\right. \tag{40}$$

This model takes into account the slope variation, change of section and the friction due to roughness on the inner wall of the pipe. This system was formally introduced by the author in [7] and [3] in the context of unsteady mixed flows in closed water pipes assuming the motion by slices that we have now justified here with the friction term.

We have proposed a finite volume discretisation of the free surface model introducing a new kinetic solver in [2, 4] based on the kinetic scheme of Perthame and Simeoni [12]. We have also proposed a new well-balanced VFRoe scheme [1]. These numerical schemes have been validated in [4] in a channel with varying width on a trans-critical steady state with shock. Several test cases have been passed with success through comparison with an exact solution or a code to code comparison, see for instance [1, 2].

4. Conclusions and perspectives

Finally, we have performed an asymptotic analysis of the three-dimensional incompressible Navier-Stokes equation with a general wall-law conditions including friction and free surface boundary conditions in the shallow water limit. We have considered the three-dimensional incompressible hydrostatic approximation with friction boundary conditions and free surface boundary conditions and we have integrated these equations along the Ω sections to get the one-dimensional free surface model. In particular, we have shown that the free surface model (40) is an approximation of $O(\varepsilon)$ of the hydrostatic approximation (20)–(22) and therefore of the three-dimensional incompressible Navier-Stokes equations (10)-(12). Except the three-dimensional model reduction to a one-dimensional one, we have shown how to integrate correctly a general friction law into the model derivation. The next step and the work in progress will consist in studying the rigorous limit.

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